ANTOINE DEZA, McMaster University, Hamilton, ON

More colourful simplices

A point $p \in R^d$ has *simplicial depth* q relative to a set S if it is contained in q closed simplices generated by $(d+1)$ sets of S. More generally, we consider *colourful simplicial depth*, where the single set S is replaced by $(d+1)$ sets, or colours, $\mathbf{S}_1, \ldots, \mathbf{S}_{d+1}$, and the *colourful* simplices containing p are generated by taking one point from each set. Assuming that the convex hulls of the $\mathbf{S}_i$'s contain p in their interior, Bárány's colourful Carathéodory's Theorem (1982) shows that p must be contained in some colourful simplex. We are interested in determining the minimum number of colourful simplices that can contain p for sets satisfying these conditions. That is, we would like to determine $\mu(d)$, the minimum number of colourful simplices drawn from $\mathbf{S}_1, \ldots, \mathbf{S}_{d+1}$ that contain $p \in R^d$ given that $p \in \mathbf{int}\big(\mathrm{conv}(\mathbf{S}_i)\big)$ for each i. Without loss of generality, we assume that the points in $\bigcup_i \mathbf{S}_i \cup \{p\}$ are in general position. The quantity $\mu(d)$ was investigated in [3], where it is shown that $2d \leq \mu(d) \leq d^2+1$, that $\mu(d)$ is even for odd d, and that $\mu(2)=5$. This paper also conjectures that $\mu(d)=d^2+1$ for all $d \geq 1$. Subsequently, Bárány and Matoušek (2007) verified the conjecture for $d=3$ and provided a lower bound of $\mu(d) \geq \max(3d, \lceil \frac{d(d+1)}{5} \rceil)$ for $d \geq 3$, while Stephen and Thomas (2008) independently provided a lower bound of $\mu(d) \geq \lfloor \frac{(d+2)^2}{4} \rfloor$. We show that for $d \geq 1$, we have $\mu(d) \geq \lceil \frac{(d+1)^2}{2} \rceil$. This strengthens the previously known lower bounds for all $d \geq 4$.

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