Algebraic Group Actions and Invariant Theory
Actions algébriques des groupes et théorie des invariants
(Org: **Eddy Campbell** (Memorial), **Juanjun Chuai** (Memorial) and/et **David Wehlau** (RMC; Queen's))

BRAM BROER, Université de Montréal
Modules of covariants in modular invariant theory

Let the finite group G act linearly on two vector spaces V and M over an arbitrary field k. The invariant ring $k[V]^G$ has as module $(k[V] \otimes_k M)^G$, called the module of covariants of type M. We study some of the properties of these modules, extending classical results form the non-modular situation to the general. As application, we get results on extensions of invariant rings.

For example, we give a generalization of a theorem of Serre's, saying that if $k[V]$ is free over $k[V]^G$, then G is generated by reflections. Put W for the (normal) subgroup of reflections in G on V and suppose $H < G$ is a subgroup such that $k[V]^G \subset k[V]^H$ is a free extension. Then $G = WH$, i.e., G is generated by H and the reflections in G. Furthermore, multiplication induces an isomorphism

$$k[V]^W \otimes_{k[V]^G} k[V]^H \to k[V]^{H \cap W},$$

hence $k[V]^W \subset k[V]^{W \cap H}$ is also a free extension and $k[V]^{H \cap W}$ is generated by W-invariants and H-invariants.

This is a report on joint work with Jianjun Chuai.

JIANJUN CHUAI, Memorial University of Newfoundland, St. John's, NL, A1C 5S7
Invariant Theory for Sub-representations

Let $G \subseteq \mathrm{GL}(V)$ be a finite linear group over a field F and let $W \subseteq V$ be an FG-submodule. In this paper, we study the relationship between the invariant rings $F[V]^G$ and $F[W]^G$. In particular, we give some necessary and sufficient conditions for the natural map $F[V]^G \mapsto F[W]^G$ to be surjective.

This is joint work with Eddy Campbell.

EMILIE DUFRESNE, Uni. Heidelberg, 69120 Heidelberg, Germany
Well-behaved separating algebras

The study of separating invariants has become quite popular in the recent years. For finite groups, a separating algebra is a subalgebra which separates the orbits. In this talk, we prove that there can exist polynomial separating algebras only when the group is generated by reflections. We thus generalize the classical result of Serre that only reflection groups may have a polynomial ring of invariants. We also show that separating algebras can be complete intersection only when the groups is generated by bireflections. We end with results on the Cohen–Macaulay property of separating algebras.

ALEX DUNCAN, University of British Columbia, 2329 West Mall, Vancouver, BC
Finite Groups of Essential Dimension 2

The essential dimension of an algebraic group is a numerical invariant which, loosely speaking, measures the number of parameters required to describe any of its actions. I will discuss essential dimension and the related concept of versal varieties. The classification of all finite groups of essential dimension 2 over the complex numbers is made possible by studying Manin and Iskovskikh's classification of minimal rational G-surfaces.

JONATHAN ELMER, RWTH Aachen, Templergraben 55, Aachen 52062, Germany
Depth and cohomological detection in invariant theory

Let G be a finite group and k a field. If the characteristic of k divides the order of G, then the ring of invariants $k[V]^G$ is usually not Cohen–Macaulay. One can often learn much about the structure of $k[V]^G$ by studying the annihilator ideals in the $k[V]^G$-modules $H^i(G, k[V])$ for $i > 0$. In this talk we will show how detection conditions on $H^i(G, k[V])$ give upper bounds for the depth of $k[V]^G$. We will also give necessary and sufficient conditions, based on cohomology, for the depth of $k[V]^G$ to be as small as possible.

JORGE FERREIRA, University of Kent, Canterbury, United Kingdom
On Invariant Rings of Sylow Subgroups of Classical Groups

One of the main interests in Modular Invariant Theory of Finite Groups is to determine presentations of the invariant ring in terms of generators and relations. In this talk we will show how to construct a presentation for the invariant ring of the Sylow p-subgroups of the unitary groups $\mathrm{GU}(3, \mathbb{F}_{q^2})$ and $\mathrm{GU}(4, \mathbb{F}_{q^2})$. We will also discuss how one might generalize these results.

PETER FLEISCHMANN, University of Kent
On inhomogeneous modular invariants of finite groups

Let k be a field and let G be a finite group. We study ungraded, commutative k-algebras R on which G acts by k-algebra automorphisms rendering R a projective kG-module. Such *projective* $k - G$*-algebras* and their invariants have a beautiful structure theory and they arise in invariant theory in the study of certain localisations.

In the case of p-groups in characteristic p, we describe the algebra D_k, which is a generator in the category of commutative, projective $k - P$-algebras, and we give explicit generators and relations for the invariant ring D_k^P. We also define and describe *simple cyclic* projective $k - P$-algebras, which include the Galois extensions of k, and *universal* projective $k - P$-algebras, from which all the others can be constructed by forming quotients and "extending invariants".

This is joint work with my colleague C. F. Woodcock (Kent).

GENE FREUDENBURG, Western Michigan University, Kalamazoo, MI 49008, USA
Additive group actions associated to derivations of $R[X, Y, Z]$ with a slice

This talk features a simple family of locally nilpotent R-derivations of $R[X, Y, Z]$ with a slice, where $R = \mathbb{C}[a, b]$. Equivalently, this is a family of $\mathbb{G}_a$-actions on $\mathbb{A}^5$ such that $\mathbb{A}^5 = V \times \mathbb{A}$, where V is the variety defined by the algebra of invariants, and $\mathbb{G}_a$ acts by translation. We show that V is an $\mathbb{A}^2$-fibration over $\mathbb{A}^2$, but it is unknown whether this is a trivial fibration. Note that V has the form $\mathrm{Spec}(B/sB)$, where $B = R[X, Y, Z]$ and $s \in B$ is the corresponding slice. We give a method for finding $f \in B$ of degree smaller than s such that B/fB and B/sB are isomorphic as fibrations. However, it is not known whether f is a slice for any locally nilpotent derivation of B. These examples are motivated by the Vénéreau polynomials $v \in L = \mathbb{C}[x, y, z, u]$. It was shown by the author that, if $K = \mathbb{C}[x, v]$, then $L[t] = K[X, Y, Z]$. The main idea is to study d/dt as a K-derivation of $K[X, Y, Z]$.

JULIA HARTMANN, RWTH Aachen University, Templergraben 55, 52062 Aachen, Germany
A local global principle for algebraic group actions and applications

Patching techniques originally used in inverse Galois theory are based on factorization theorems for invertible matrices. This talk generalizes these factorization results from Gl_n to rational linear algebraic groups. As a consequence, one obtains a local global principle for homogeneous spaces under such groups. One application is a new proof of the recent result of Parimala and Suresh on the maximal dimension of anisotropic quadratic forms over p-adic function fields (u-invariant). The same approach yields results on the period-index problem for central simple algebras.

JIA HUANG, School of Mathematics, University of Minnesota, Minneapolis, MN 55455, USA
Linear Sparsity Groups and Their Polynomial Invariants

We define a sparsity pattern over a field F as a map $\sigma\colon [n] \times [n] \to \mathcal{P}(F) \setminus \{\emptyset\}$ where $\mathcal{P}(F)$ is the power set consisting of all subsets of F. We show that all matrices in $\mathrm{GL}(n, F)$ subordinate to the pattern σ form a group

$$\mathrm{GL}_\sigma(n, F) = \{M = [a_{ij}]_{n\times n} \in \mathrm{GL}(n, F) : a_{ij} \in \sigma(i, j)\}$$

if and only if σ satisfies certain completeness conditions. We prove that the ring of invariants of each finite sparsity subgroup is a polynomial algebra. This result generalizes Dickson's Theorem on invariants of $\mathrm{GL}(n, F_q)$ as well as many other examples. By viewing $\mathrm{GL}(n, F)$ as a Lie group we obtain another interpretation of an arbitrary sparsity pattern which can be generalized to Chevalley groups and reductive algebraic groups.

GREGOR KEMPER, Technische Universität München
The computation of invariant fields

The talk presents an algorithm for the computation of invariant fields that applies to a surprisingly general situation: it works for every action, given by a morphism, of an algebraic group on an irreducible variety. The ideas also lead to a constructive version of a theorem of Rosenlicht, which says that almost all orbits can be separated by rational invariants.

MARTIN KOHLS, Technische Universität München, Zentrum Mathematik–M11, Boltzmannstr. 3, 85748 Garching, Germany
The depth of invariants of (infinite) algebraic groups

The depth of invariant rings $K[V]^G$ of finite groups has been a topic in invariant theory for almost 30 years now. The depth is bounded above by the dimension of the invariant ring, and the difference between both numbers is called the Cohen–Macaulay defect. This is the minimal length of a free resolution of $K[V]^G$ over a subalgebra A generated by a homogeneous system of parameters. Thus, the Cohen–Macaulay defect is a measure for the structural complexity of $K[V]^G$. So far, almost nothing has been known about the depth of invariants of (infinite) algebraic groups. Our main result is, that for any reductive group G, which is not linearly reductive, there exists a faithful G-module V such that the Cohen–Macaulay defect of the vector invariants $K[V^{\oplus k}]^G$ is at least $k - 2$ for all k.

NICOLE LEMIRE, University of Western Ontario
Upper bounds for the Essential Dimension of a Moduli Stack of SL_n Bundles over a Curve

In joint work with Ajneet Dhillon, we find upper bounds for the essential dimension of various moduli stacks of SL_n bundles over a curve. When n is a prime power, our calculation gives the precise value of the essential dimension of the stack of stable bundles.

UGUR MADRAN, Izmir University of Economics, Izmir, Turkey
On the invariants of non-abelian p-groups

Let $\rho\colon G \hookrightarrow \mathrm{GL}(V)$ be a representation of a finite group G over a finite field, $\mathbb{F}$. The main object of study is the invariant ring $\mathbb{F}[V]^G$ where the action of G on V^* is induced by the representation. Describing $\mathbb{F}[V]^G$ is of fundamental importance and when $|G| \notin \mathbb{F}^\times$ only a little is known. There is no analog of Noether bound, depending only on the group order where Noether bound gives the maximum degree of a polynomial in a minimal generating set.

Invariants of upper triangular unipotent groups are known to be polynomials. Since any p-group is contained in some Sylow p-subgroup, without loss of generality (up to a change of basis), any p-groups can be assumed to be a subgroup of upper triangular unipotent matrices. Hence, understanding invariants of p-groups will enable us to understand modular invariants in detail. Cyclic groups of prime power order p^α are studied in the literature, but only a little is known for nonabelian groups.

In this study, we will restrict our attention to finding invariants of a nonabelian p-group of order p^3 and of exponent p. The first such nontrivial representation occurs in dimension 4. The underlying field will be the prime field and $p > 3$.

ANNETTE MAIER, RWTH Aachen, Templergraben 55, 52062 Aachen, Germany
Frobenius modules and Galois groups in positive characteristic

It is an open question which finite groups occur as Galois groups over function fields over finite fields. So-called Frobenius modules have proved useful to study this question in the case of finite groups of Lie type. The method makes use of upper and lower bound criteria for the Galois group of a Frobenius module due to Matzat, and it deploys the structure of the underlying linear algebraic group. So far, all classical groups and the exceptional groups of small rank have been realized as Galois groups in this way. It is also possible to derive polynomials for the Galois extension which are in most cases of a particularly simple form.

MUFIT SEZER, Bilkent University, Ankara, Turkey
Constructing modular separating invariants

Consider a finite dimensional modular representation V of a cyclic group of prime order p. Two points in V that are in different orbits can be separated by an homogeneous invariant polynomial that has degree one or p and that involves variables from at most two summands in the dual representation. I will also talk about some lexsegment and Gotzmann ideals in invariant theory.

JIM SHANK, University of Kent, Canterbury, CT2 7NF, United Kingdom
On the ring of invariants of the third symmetric power representation of $\mathrm{SL}(2,p)$

I will describe an explicit finite generating set for the ring of invariants for the third symmetric power representation of $\mathrm{SL}(2,p)$. The proof that the described invariants generate relies on the construction of an infinite SAGBI basis and a Hilbert series computation.

The presentation will be based on work with my PhD student, Ashley Hobson.

PETER SYMONDS, University of Manchester
Regularity of invariant rings

We sketch a proof of the conjecture that the invariants of a finite group G acting on a polynomial ring in n variables over a finite field are generated in degrees at most $n(|G|-1)$ (for $n, |G|>1$).

DAVID WEHLAU, Royal Military College of Canada
First main theorems for $\mathrm{SL}_2(\mathbb{F}_p)$ *and* C_p

I will describe a proof of the first main theorem for two-dimensional modular representations of the cyclic group of order p. I will explain how this result may be used to construct a generating set for the ring of vector invariants $\mathbb{F}[mV]^{\mathrm{SL}_2(\mathbb{F})}$, where $\mathbb{F}$ is any field of characteristic p and V is the defining representation of $\mathrm{SL}_2(\mathbb{F})$.

This is joint work with Eddy Campbell (Memorial University) and Jim Shank (University of Kent).