Complex Function Theory
Théorie des fonctions complexes
(Org: **Ian Graham** (Toronto) and/et **Eric Schippers** (Manitoba))

ROGER BARNARD, Texas Tech University, Lubbock, Texas

Iceberg-type problems in two dimensions

We consider the complex plane $\mathbb{C}$ as a space filled with two different media, separated by the real axis $\mathbb{R}$. Let H denote the upper half-plane. For a planar body E, the iceberg-type problem is to estimate characteristics of the invisible part $E \setminus H$ from the characteristics of the whole body E and its visible part $E \cap H$.

In this talk, we outline the methods we use to determine the maximal draft of E as an explicit function of the logarthmic capacity of E and the area of $E \cap H$.

TOM BLOOM, University of Toronto, 40 St. George St., Toronto

Random polynomials and pluripotential theory

I will report on results giving the expected distribution of zeros of certain ensembles of random polynomials in one and several complex variables as the equilibrium measure of compact sets.

This is joint work with B. Shiffman.

ANDRÉ BOIVIN, University of Western Ontario

Weighted Hardy spaces for the unit disc: Approximation properties

We will state some basic properties of the weighted Hardy space for the unit disc with the weight function satisfying Muckenhoupt's (A^p) condition $(1 < p < \infty)$. Approximation properties in that space of the system of rational functions $e_k(z) = \dfrac{1}{(2\pi i)(1 - \overline{a}_k z)}$, where $\{a_k\}$ is a sequence satisfying the Blaschke condition $\sum_{k=1}^{\infty}(1 - |a_k|) < \infty$, will then be discussed.

MARITZA BRANKER, Niagara University, Lewiston, NY

Weighted pluripotential theory on Kahler manifolds

This talk will describe an extension of weighted pluripotential theory, based on quasiplurisubharmonic functions on compact Kahler manifolds.

The work described is a collaborative effort with M. Stawiska.

RICHARD FOURNIER, Dawson College–CRM, 3040 Sherbrooke West, Montreal

Asymptotics of the Bohr radius for polynomials of fixed degree

We obtain a characterization and precise asymptotics of the Bohr radius for the class of complex polynomials in one variable. Our work is based on the notion of bound-preserving operators.

DAVID HERRON, University of Cincinnati
Euclidean Quasiconvexity

A metric space is quasiconvex provided it is bilipschitz equivalent to a length space: each pair of points can be joined by a rectifiable path whose length is comparable to the distance between its endpoints.

We consider a closed set in Euclidean n-space and ask when is its complement quasiconvex. In dimension $n = 2$, a complete description is available, at least for closed sets with finitely many components. In general, there are sufficient conditions that such a complement be quasiconvex; one such condition is that the set have zero $(n-1)$-dimensional Hausdorff measure.

We exhibit, for each dimension d in $[n-1, n]$, a compact totally disconnected set with positive finite d-measure whose complement is quasiconvex. On the other hand, we also construct a compact totally disconnected set with non-zero $(n-1)$-measure whose complement fails to be quasiconvex.

Joint work with Hrant Hakobyan.

DAN JUPITER, Department of Systems Biology and Translational Medicine, College of Medicine, Texas A&M Health Science Center
Weak Stein Neighbourhood Bases

It is of interest to understand whether the closure of a pseudoconvex domain in $\mathbb{C}^n$ has a neighbourhood basis of pseudoconvex domains. Another question of interest is to understand when holomorphic functions on a pseudoconvex domain can be approximated by holomorphic functions on a larger set. We discuss some aspects of the relationship between these two types of approximation properties.

GABRIELA KOHR, Babes-Bolyai, Romania

DANIELA KRAUS, University of Würzburg
Critical points of inner functions, nonlinear partial differential equations, and an extension of Liouville's theorem

Theorem 1 *Let $\{z_j\} \subseteq \mathbb{D}$ be a Blaschke sequence. Then there exists a Blaschke product with critical points $\{z_j\}$ (counted with multiplicity) and no others.*

The case of finitely many critical points has been proved earlier by Heins 1962, Wang & Peng 1979, Bousch 1992 and Zakeri 1996 (topological proofs) and by Stephenson 2005 (discrete methods). It has found applications in complex dynamics by Milnor.

The proof of Theorem 1 is based on an extension of Liouville's classical representation theorem for solutions of the partial differential equation $\Delta u = 4e^{2u}$ combined with methods from nonlinear elliptic PDE. Our work is closely related to the Berger–Nirenberg problem in differential geometry.

Joint work with Oliver Roth.

JAVAD MASHREGHI, Laval
Integral representations of the derivatives of functions in $\mathcal{H}(b)$

Let $H^p(\mathbb{C}_+)$ stand for the Hardy space of the upper half plane $\mathbb{C}_+$, and for $\varphi \in L^\infty(\mathbb{R})$, let T_φ stand for the Toeplitz operator defined on $H^2(\mathbb{C}_+)$ by

$$T_\varphi(f) := P_+(\varphi f), \quad \big(f \in H^2(\mathbb{C}_+)\big),$$

where P_+ denotes the orthogonal projection of $L^2(\mathbb{R})$ onto $H^2(\mathbb{C}_+)$. Then, for $\varphi \in L^\infty(\mathbb{R})$, $\|\varphi\|_\infty \le 1$, the de Branges–Rovnyak space $\mathcal{H}(\varphi)$, associated to φ, consists of those $H^2(\mathbb{C}_+)$ functions which are in the range of the operator $(\mathrm{Id} - T_\varphi T_{\overline{\varphi}})^{1/2}$. It is a Hilbert space when equipped with the inner product

$$\langle (\mathrm{Id} - T_\varphi T_{\overline{\varphi}})^{1/2} f, (\mathrm{Id} - T_\varphi T_{\overline{\varphi}})^{1/2} g \rangle_\varphi = \langle f, g \rangle_2,$$

where $f, g \in H^2(\mathbb{C}_+) \ominus \ker(\mathrm{Id} - T_\varphi T_{\overline{\varphi}})^{1/2}$. In particular, if b is an inner function, then $(\mathrm{Id} - T_b T_{\overline{b}})$ is an orthogonal projection and $\mathcal{H}(b)$ is a closed (ordinary) subspace of $H^2(\mathbb{C}_+)$ which coincides with the so-called model spaces $K_b = H^2(\mathbb{C}_+) \ominus b H^2(\mathbb{C}_+)$.

We give some integral representations for the boundary values of derivatives of functions of the de Branges–Rovnyak spaces $\mathcal{H}(b)$, where b is an extreme point of the unit ball of $H^\infty(\mathbb{C}_+)$.

DAVID MINDA, University of Cincinnati, Cincinnati, Ohio
Geometric variations of Schwarz's Lemma

The classical version of Schwarz's Lemma deals with holomorphic self-maps of the unit disk $\mathbb{D}$ that fix the origin; the extremal functions for Schwarz's Lemma are rotations about the origin. We consider holomorphic maps of $\mathbb{D}$ into a region Ω that satisfies some geometric property that holds for the unit disk. For example, Ω has diameter at most 2. There are regions of diameter 2 that are not contained in a disk of radius 1, so this case properly contains the classical framework. Landau and Toeplitz in considered this situation in 1907. Other geometric conditions on Ω involve the area, length of the boundary and higher-order diameters, including the transfinite diameter. In all cases we obtain sharp analogs of the classical Schwarz Lemma and identify the extremal functions.

JERRY MUIR JR., University of Scranton, Scranton, PA
Preservation of geometric properties by a class of extension operators

We consider operators that extend locally univalent mappings of the unit disk Δ of $\mathbb{C}$ to locally biholomorphic mappings of the Euclidean unit ball B of $\mathbb{C}^n$. For such an operator Φ, we ask whether $\Phi(f)$ is a convex (resp. starlike) mapping of B whenever f is a convex (resp. starlike) mapping of Δ or whether $e^t \Phi\big(e^{-t} f(\cdot, t)\big)$, $t \ge 0$, is a Loewner chain on B whenever $f(\cdot, t)$, $t \ge 0$, is a Loewner chain on Δ. Answers will be provided for a class of operators that are perturbations of the well known Roper–Suffridge extension operator.

RAJESH PEREIRA, University of Saskatchewan
Inequalities in the Analytic Theory of Polynomials

We show how techniques in matrix theory and majorization can be used to derive inequalities relating the zeros and critical points of a polynomial. These inequalities stengthen known results such as the Gauss–Lucas Theorem and Mahler's inequality. We will also present some partial results and conjectures on a Kroo–Pritsker type inequality between the Bombieri norm of a polynomial and the product of the Bombieri norms of its linear factors.

JOHN PFALTZGRAFF, University of North Carolina, Chapel Hill, NC 27599-3250, USA
Conformal mapping of multiply connected slit domains

The first general formula for a Schwarz–Christoffel mapping of a canonical domain of connectivity $m > 2$ onto a conformaly equivalent polygonal domain appears in work with DeLillo and Elcrat [DEP,04]. Construction of the mapping and its formula uses infinite sequences of iterated reflections in circles, repeated use of the reflection principle and invariance of the preSchwarzian to obtain an infinite product representation of the derivative of the map and an integral formula for the mapping function. The method can be interpreted as a form of the "method of images" in electrostatics.

The problem of implementing the formula numerically and graphically is pursued in [DDEP,06]. Developing a robust code and a complete, easy to apply procedure remains a challenging problem. In current work with DeLillo, Driscoll and Elcrat, interesting special features have appeared when the target domains are certain canonical slit domains. For example, the direct construction of a formula for the mapping function that produces $f(z)$ explicitly without requiring an integration of the derivative.

Remark The results in [DEP,04] were presented by Elcrat and Pfaltzgraff in 2003 at international meetings ICIAM in Australia and AMS–RSME, Seville, Spain.

References

[DEP,04] T. DeLillo, A. Elcrat and J. Pfaltzgraff, *Schwarz–Christoffel mapping of multiply connected domains.* J. Anal. Math. **94**(2004), 17–47.

[DDEP,06] T. DeLillo, T. Driscoll, A. Elcrat and J. Pfaltzgraff, *Computaton of multiply connected Schwarz–Christoffel maps for exterior domains.* Comput. Methods Funct. Theory **6**(2006) 301–315.

DAVID RADNELL, American University of Sharjah, United Arab Emirates
Interactions between conformal field theory and Teichmueller spaces

Conformal Field Theory (CFT) arose in physics as a special class of two-dimensional quantum field theories. The mathematics of CFT requires the study of Riemann surface whose boundary components are parameterized. The moduli space of these rigged Riemann surfaces arises naturally in the mathematical description. The sewing of two Riemann surfaces by identifying the boundary components is a fundamental operation.

We have recently applied results from Teichmueller theory, such as conformal welding, to CFT. In particular, the sewing operation was shown to be holomorphic.

In on-going work, these results and ideas from CFT are used to give a new structure to the infinite-dimensional Teichmueller space of bordered Riemann surfaces. We show that this Teichmueller space is a complex fiber space over the finite-dimensional Teichmueller space of punctured surfaces. The fibers are spaces of conformal maps with quasiconformal extensions that are closely related to the universal Teichmueller space.

This introductory talk will overview the new and rewarding interplay between these fields.

This is joint work with E. Schippers.

THOMAS RANSFORD, Université Laval, Québec (QC), G1K 7P4
Computation of capacity

I shall describe a method for computing the logarithmic capacity of a compact plane set. The method yields upper and lower bounds for the capacity. If the set has the Hölder continuity property, then these bounds converge to the value of the capacity. I shall discuss several examples, including the Cantor middle-third set, for which we estimate $c(E) \approx 0.220949102189507$.

Joint work with Jérémie Rostand.

OLIVER ROTH, University of Würzburg
On the isolated singularitites of a class of subharmonic functions

We extend a classical result proved by Nitsche in 1957 about the isolated singularities of the solutions of the Liouville equation $\Delta u = 4e^{2u}$ to solutions of the Gaussian curvature equation $\Delta u = -4\kappa(z)e^{2u}$ where κ is a strictly negative Hölder continuous function. This yields growth and regularity theorems for strictly negatively curved conformal Riemannian metrics close to their singular points which complement the corresponding existence-and-uniqueness results due to Heins, Troyanov, McOwen and others.

Joint work with Daniela Kraus.

STEPHAN RUSCHEWEYH, Department of Mathematics, Wuerzburg University, 97074 Wuerzburg, Germany
Universally convex univalent functions

A function f analytic in the slit domain $\mathbb{C}[1, \infty]$ is called universally convex if it maps every circular domain containing the origin but not the point 1 univalently onto a convex domain. We give a complete characterization of those functions in terms of Hausdorff moment sequences, and show that this set is closed under convolutions (Hadamard product). Some generalisations are also mentioned.

Joint work with L. Salinas, Valparaíso, and T. Sugawa, Hiroshima.

ALEXANDER SOLYNIN, Texas Tech University, Department of Mathematics and Statistics, Lubbock, TX 79409, USA
Hyperbolic convexity and the analytic fixed point function

We will discuss properties of the *analytic fixed point function* introduced recently by D. Mejia and Ch. Pommerenke. In particular, we solve one of the problems raised by D. Mejía and Ch. Pommerenke by showing that the analytic fixed point function is hyperbolically convex in the unit disc. We also prove some extremal properties of such functions related to mappings from the unit disk onto symmetric Riemann surfaces.

TED SUFFRIDGE, Department of Mathematics, University of Kentucky, Lexington, KY 40506
Invariant mappings on the ball and extremal problems

The concept of "linear invariant family" was introduced by Pommerenke in his 1964 paper in Mathematische Annalen. A family $\mathcal{F}$ of functions that are analytic on the unit disk and normalized by $f(0) = 0$, $f'(0) = 1$ with $f'(z) \neq 0$ when $|z| < 1$ is linear invariant provided that the function $K_\varphi f \in \mathcal{F}$ whenever $f \in \mathcal{F}$. Here, φ is a holomorphic automorphism of the unit disk, and $K_\varphi f$ is obtained by forming the composition $f \circ \varphi$ and normalizing the result. The functions that have the property $K_\varphi f = f$ for certain automorphisms φ are of particular interest and in fact the solution of many extremal problems on a family $\mathcal{F}$ is one of these "invariant" functions. We discuss the extension of these ideas to mappings $f \colon B \to \mathbb{C}^n$, where B is the Euclidean ball in $\mathbb{C}^n$, and in fact characterize the invariant mappings for given linear invariant familes of mappings, in a theorem that gives a procedure for constructing all such mappings.

This is joint work with J. A. Pfaltzgraff.

DROR VAROLIN, Stony Brook University
Non-negative Hermitian polynomials and quotients of squared norms

Hilbert's 17th problem asks whether any non-negative polynomial can be written as a sum of squares of rational functions. While a positive answer was established by Artin, there is no known way to construct the rational functions. In this talk we describe our solution of a Hermitian analog of Hilbert's 17th problem posed by John D'Angelo about 14 years ago.

BROCK WILLIAMS, Texas Tech University, Lubbock, Texas
Circle Packings, Welding Operations, and Applications

We will discuss various welding operations in the plane and on Riemann surfaces and their circle packing analogues. For example, circle packings can be used to approximate classical quasisymmetric welding, such as arises in the work of Radnell and Schippers in string theory. We will also consider other welding-type deformations included Thurston's earthquakes and McMullen's complex earthquakes.

This is joint work with Roger Barnard, J'Lee Bumpus, and Eric Murphy.