Report of the Forty Second Canadian Mathematical Olympiad 2010

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The Canadian Mathematical Olympiad (CMO) is an annual national mathematics competition sponsored by the Canadian Mathematical Society (CMS) and is administered by the Canadian Mathematical Olympiad Committee (CMO Committee), a sub-committee of the Mathematical Competitions Committee. The CMO was established in 1969 to provide an opportunity for students who performed well in various provincial mathematics competitions to compete at a national level. It also serves as preparation for those Canadian students competing at the International Mathematical Olympiad (IMO).

Students qualify to write the CMO by earning a sufficiently high score on the Sun Life Financial Canadian Open Mathematics Challenge (COMC). This year, students with the top 68 COMC scores were invited outright to write the CMO. Approximately 100 others, next in rank, were invited to send solutions to a Repechage set of 10 problems posted online within a week to the University of Waterloo. Twenty four students were invited from the Repechage. The CMO Qualifying Repechage that in the past few years has been run on experimental basis is now a well-established competition with the goal of selecting additional students for the CMO. I am grateful to Ian VanderBurgh for setting this up and assembling a team of markers consisting of Serge D'Alessio, Fiona Dunbar, Mike Eden, Barry Ferguson, Steve Furino, Judith Koeller, Jen Nissen, J.P. Pretti, Ian VanderBurgh and Troy Vasiga, to go through the 82 scripts received.

The Society is grateful for the support from **Sun Life Financial** and the other sponsors listed on the previous page.

I am very grateful to the CMO Committee members for submitting the problems, reviewing the test and marking the solutions: Andrew Adler, Edward Barbeau, Jason Bell, Julia Gordon, Robert Morewood, Zinovy Reichstein, Naoki Sato, Jozsef Solymosi and Adrian Tang.Thanks also go to Thomas Griffiths for reviewing the final paper and to Joseph Khoury for translating the problems into French. Finally, I'd like to thank Susan Latreille, Laura Alyea and the Executive Director Johan Rudnick for the hard work done at the CMS headquarters.

Kalle Karu, Chair
Canadian Mathematical Olympiad Committee

The 42nd Canadian Mathematical Olympiad was written on Wednesday, March 24, 2010. A total of 98 sets of solutions were received. Of these, 77 were eligible for official prizes. There were 80 students from schools in Canada and 18 from abroad. Six Canadian provinces were represented, with the number of contestants as follows:

AB (7) BC (19) MB (1) ON (48) QC (4) SK (1)

The 2010 CMO consisted of five questions, each marked out of seven. The maximum score obtained by the winner was 30 marks. The official contestants were grouped into four divisions according to their scores as follows:

Division	Range of Scores	# of Students
I	19-30	8
II	15-18	16
III	9-14	27
IV	0-8	47

FIRST PRIZE – Sun Life Financial Cup - $2000

Alex Song
Vincent Massey Secondary School, Windsor, ON

SECOND PRIZE - $1500

James Rickards
Colonel By Secondary School, Greely, ON

THIRD PRIZE - $1000

Jonathan Zung
University of Toronto Schools, Toronto, ON

HONOURABLE MENTIONS - $500

Robin Cheng
Pinetree Secondary School, Coquitlam, BC

Zhi Qiang Liu
Don Mills Collegiate Institute, Toronto, ON

Chen Sun
A.B. Lucas Secondary School, London, ON

Jixuan Wang
Don Mills Collegiate Institute, Toronto, ON

Yuqi Zhu
University Hill Secondary School, Vancouver, BC

Division 1 19-30

Alex Song	Vincent Massey S.S.	ON
James Rickards	Colonel By S.S.	ON
Jonathan Zung	University of Toronto Schools	ON
Robin Cheng	Pinetree S.S.	BC
Zhi Qiang Liu	Don Mills C.I.	ON
Chen Sun	A.B. Lucas S.S.	ON
Jixuan Wang	Don Mills C.I.	ON
Yuqi Zhu	University Hill S.S.	BC

Division 2 15-18

Yaroslav Babich	Sir Winston Churchill H.S.	AB
Brian Bi	Woburn C.I.	ON
Zhangchi Chen*	Suzhou H.S., China	
Calvin Deng	Enloe High School	NC
James Duyck	Vincent Massey S.S.	ON
Neil Gurram	ICAE	MI
Soroosh Hemmati	Western Canada H.S.	AB
Heinrich Jiang	Vincent Massey S.S.	ON
Kevin Li	A&M Consolidated H.S.	TX
Mariya Sardarli	Strathcona Comp. H.S.	AB
Susan Sun	West Vancouver S.S.	BC
Zijian Yao*	Lester Pearson College	BC
Bill Ye	Olympiads School	ON
Pei Jun Zhao	London Central S.S.	ON
Kevin Zhou	Woburn C.I.	ON
Jonathan Y Zhou	Pinetree S.S.	ON

Division 3 9-14

Joshua Alman	University of Toronto Schools	ON
Ram Bhaskar*	ICAE	MI
Sifan Bi*	Sir John A. Macdonald S.S.	ON
Matthew Brennan	Upper Canada College School	ON
Richard Chen	Sir John A. Macdonald C.I.	ON
Liqing Ding	Branksome Hall	ON
Kun Dong*	Sir William Mulock S.S.	ON
Yale Fan*	Valley C.S.	OR
Jun Hou Fung*	Can. Int'l School of Hong Kong	
Jiayue Gao	Sir Winston Churchill S.S.	BC
Fang Guo	Richmond Hill H.S.	ON
Ursula Anne Lim	Burnaby North S.S.	BC
David Siqi Liu	Vincent Massey S.S.	ON
Jackie Liu	Sir Winston Churchill S.S.	BC
Anupa Murali	Bishop Brady H.S.	NH
Chang Sun Park	Magee S.S.	BC
Zhongwu Shi*	Suzhou H.S., China	
Hunter Spink	Western Canada H.S.	AB
Richard Wang	Sir Winston Churchill S.S.	BC
Zexuan Wang	A.Y. Jackson S.S.	ON
Shen Wang	Lord Byng S.S.	BC
Tongbin Wu	White Oaks S.S.	ON
Yu Wu	Agincourt C.I.	ON
Allen Yang	Cary Academy	NC
Steven Yu	Pinetree S.S.	BC
Joe Zeng	Don Mills C.I.	ON
Kaiven Zhou	Strathcona Comp. H.S.	AB

Division 4 0-8

Steven Chang*	ICAE	MI
Brynmor Chapman*	Valley C.S.	OR
Wonjohn Choi*	St. Francis Xavier S.S.	ON
Alexander Cowan	Marianopolis College	QC
Yuchen Cui	Martingrove C.I.	ON
Aden Dong	A.Y. Jackson S.S.	ON
Brandon Ewonus	St. Michael's Univ. School	BC
Harsha Gotur*	ICAE	MI
Changho Han	Bayview S.S.	ON
Louis Hong	Sir John A. Macdonald S.S.	ON
Albert Hu	Northern S.S.	ON
Sufyan Khan	Earl Haig S.S.	ON
Namhun Kim	ICAE	MI
Sung Jun Kim*	South Island S.S. Hong Kong	
David Kong	Glenforest S.S.	ON
Leo Lai	Prince of Wales S.S.	BC
Eung Bum Lee	West Vancouver S.S.	BC
Sukwan Lee	Heritage Woods S.S.	BC
Shen Li	Marianopolis College	QC
Felix Li	Univ. of Toronto Schools	ON
Albert Liao	St. John's Ravenscourt	MB
Yangsheng Liu	Dr. Norman Bethune C.I.	ON
David Lu*	ICAE	MI
Juntao Luo	Lawrence Park C.I.	ON
Tonghui Ma	Sir John A. Macdonald C.I.	ON
Tina Marie Mitre	Dawson College	QC
Ryan Peng	Centennial College	SK
Aurick Qiao	Vincent Massey S.S.	ON
Ritvik Ramkumar	Glenforest S.S.	ON
Cristina Rosu	Univ. of Toronto Schools	ON
Wen Yi Song	Semiahmoo S.S.	BC
Shai Spilberg	Vanier College	QC
Lexuan Wang	Hugh McRoberts S.S.	BC
Michael Wong	Western Canada H.S.	AB
Kaiyu Wu	Meadowvale S.S.	ON
Tian Xia	The Woodlands School	ON
Yejia Xu*	Suzhou High School	
Yung Lin Yang	Northern S.S.	ON

Mertcan Yetkin*	Tevitol H.S., Turkey	
Fan Yin	Vincent Massey S.S.	ON
Daniel Yoo	Thornhill S.S.	ON
Simon Younan	St. Francis Xavier S.S.	ON
Fangcun Yu	Sir John A. Macdonald C.I.	ON
Eric Zhan	Univ. of Toronto Schools	ON
Cyril Zhang	Don Mills C.I.	ON
Justine Zhang	Sir Winston Churchill H.S.	AB
Allen Zhang	Univ. Transition Program	BC

The students indicated by * wrote the 2010 CMO as unofficial candidates.

The Grader's Report

The 2010 CMO was marked by Andrew Adler, Jason Bell, Julia Gordon, Robert Morewood and Kalle Karu on April 10, 2010. All 95 papers received by that time were carefully marked once. Then the top 1/3 of papers with scores 13 or higher were then marked the second time.

The overall difficulty of the problems this year was very good for separating the top students. The highest 8 scores were evenly spaced in the range 19-30. This also meant that the majority of the scores were below 15 marks. The table below lists the number of points earned for each problem.

Score	Problem #1	Problem #2	Problem #3	Problem #4	Problem #5
7	2	49	7	9	3
6	1	5	3	1	0
5	2	0	21	0	0
4	7	4	3	3	1
3	19	3	7	2	2
2	12	3	9	9	1
1	42	8	6	23	5
0	10	11	17	13	32
-	3	15	25	38	54

The problems this year did not span a scale from very easy to very hard. They were mostly at the same level of difficulty, the easiest two being the Euclidean geometry problem (Problem #2) and the speed-skating problem (Problem #3). The remaining problems were equally hard, requiring knowledge of different areas, such as graphs, division of polynomials, modular arithmetic.

Problem #1. Even though appearing as the first problem, this was not the easiest one. Many students got part (a) of the problem correct, but very few correct solutions to part (b) were found. The special case of n=2^k-2 was often thought to be the only solution to part (b). Another common omission was to construct a tiling with a given value of f(n), but not prove that this is indeed the minimal tiling. Since this was the first problem, almost everybody attempted it. Some students spent a long time on it, often trying to prove a wrong conjecture.

Problem #2. There were a number of different approaches to this problem, many of them leading to correct solutions. The most popular correct solution was the one coming from considering two pairs of similar right triangles (FAP similar to PGB, and AGP similar to PBH, in the notation of the official Solution 2). Solution 1 also occurred a few times, with a few variations, but much less frequently. There was one solution (and two or three more entirely unsuccessful attempts) by computing everything in coordinates (the successful solution involved choosing reasonable coordinate axes and using the formula for a distance from a point to a given line). The most common error was the one that appears in the official Solution 2: when discussing these cyclic quadilaterals, some students overlooked the possibility that P might be close enough to A or B (on the larger arc between the two), so that the foot of the perpendicular line from P to AB is outside the circle. In this case some slightly different angles need to be considered.

Problem #3. The problem was not very hard, but very few students gave a correct answer. The most common approach was to compute the time it takes to finish the race. From this one can find that the fastest skater passes the slowest one 60 times. The most common error was to then conclude that there are 59 possibilities for the middle racer, failing to notice that if this number is not relatively prime to 60, then the race ends earlier. Two marks were taken off for this mistake, which explains the relatively large number of 5 marks. The approach given in the first official solutions was not used by any student. The alternate solution was the most often used one.

Problem #4. The main difficulty here was that many students did not know what a graph is. This possibility was discussed during problem selection and a short definition of a graph was added to the problem. This did not seem to clear the definition. Sometimes a graph was assumed to be a string of nodes and edges, sometimes a square grid. This problem was sufficiently abstract, so that doing examples with simple graphs did not lead to a proof. Applying induction was the only successful approach. A couple of students reduced the problem to solving a system of linear equations over the 2-element field, but no complete proof in this direction was given.

Problem #5. The main difficulty in problem 5 was that it was the last problem and most students ran out of time. Only 3 students fully solved it. A few more, not many, got part marks. Some tried a doomed induction on degree. The only successful approach started by expressing P(x)/Q(x) as A(x)+R(x)/Q(x). Several who tried this unfortunately assumed or even asserted that $A(x)$ has integer coefficients. Several students made partial progress by arguing more or less correctly that R(x)= 0, though the argument here was often not clear.

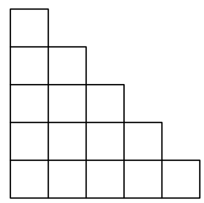
Appendix

42^{nd} Canadian Mathematical Olympiad 2010

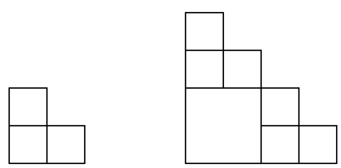
Problems and Solutions

CANADIAN MATHEMATICAL OLYMPIAD 2010
PROBLEMS AND SOLUTIONS

(1) For a positive integer n, an n-*staircase* is a figure consisting of unit squares, with one square in the first row, two squares in the second row, and so on, up to n squares in the n^{th} row, such that all the left-most squares in each row are aligned vertically. For example, the 5-staircase is shown below.



Let $f(n)$ denote the minimum number of square tiles required to tile the n-staircase, where the side lengths of the square tiles can be any positive integer. For example, $f(2) = 3$ and $f(4) = 7$.



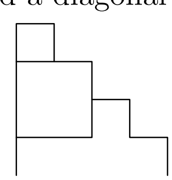
(a) Find all n such that $f(n) = n$.
(b) Find all n such that $f(n) = n + 1$.

Solution. (a) A *diagonal* square in an n-staircase is a unit square that lies on the diagonal going from the top-left to the bottom-right. A *minimal tiling* of an n-staircase is a tiling consisting of $f(n)$ square tiles.

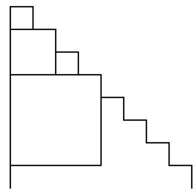
Observe that $f(n) \geq n$ for all n. There are n diagonal squares in an n-staircase, and a square tile can cover at most one diagonal square, so any tiling requires at least n square tiles. In other words, $f(n) \geq n$. Hence, if $f(n) = n$, then each square tile covers exactly one diagonal square.

Let n be a positive integer such that $f(n) = n$, and consider a minimal tiling of an n-staircase. The only square tile that can cover the unit square in the first row is the unit square itself.

Now consider the left-most unit square in the second row. The only square tile that can cover this unit square and a diagonal square is a 2×2 square tile.



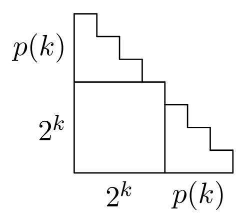
Next, consider the left-most unit square in the fourth row. The only square tile that can cover this unit square and a diagonal square is a 4×4 square tile.



Continuing this construction, we see that the side lengths of the square tiles we encounter will be 1, 2, 4, and so on, up to 2^k for some nonnegative integer k. Therefore, n, the height of the n-staircase, is equal to $1+2+4+\cdots+2^k = 2^{k+1}-1$. Alternatively, $n = 2^k - 1$ for some positive integer k. Let $p(k) = 2^k - 1$.

Conversely, we can tile a $p(k)$-staircase with $p(k)$ square tiles recursively as follows: We have that $p(1) = 1$, and we can tile a 1-staircase with 1 square tile. Assume that we can tile a $p(k)$-staircase with $p(k)$ square tiles for some positive integer k.

Consider a $p(k+1)$-staircase. Place a $2^k \times 2^k$ square tile in the bottom left corner. Note that this square tile covers a digaonal square. Then $p(k+1) - 2^k = 2^{k+1} - 1 - 2^k = 2^k - 1 = p(k)$, so we are left with two $p(k)$-staircases.

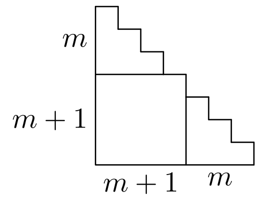


Furthermore, these two $p(k)$-staircases can be tiled with $2p(k)$ square tiles, which means we use $2p(k) + 1 = p(k+1)$ square tiles.

Therefore, $f(n) = n$ if and only if $n = 2^k - 1 = p(k)$ for some positive integer k. In other words, the binary representation of n consists of all 1s, with no 0s.

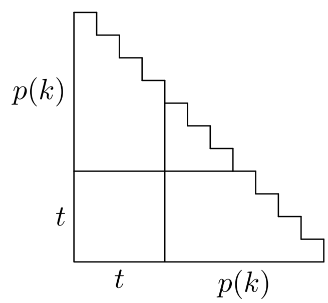
(b) Let n be a positive integer such that $f(n) = n + 1$, and consider a minimal tiling of an n-staircase. Since there are n diagonal squares, every square tile except one covers a diagonal square. We claim that the square tile that covers the bottom-left unit square must be the square tile that does not cover a diagonal square.

If n is even, then this fact is obvious, because the square tile that covers the bottom-left unit square cannot cover any diagonal square, so assume that n is odd. Let $n = 2m + 1$. We may assume that $n > 1$, so $m \geq 1$. Suppose that the square tile covering the bottom-left unit square also covers a diagonal square. Then the side length of this square tile must be $m + 1$. After this $(m+1) \times (m+1)$ square tile has been placed, we are left with two m-staircases.

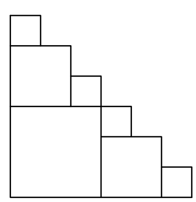


Hence, $f(n) = 2f(m)+1$. But $2f(m)+1$ is odd, and $n+1 = 2m+2$ is even, so $f(n)$ cannot be equal to $n+1$, contradiction. Therefore, the square tile that covers the bottom-left unit square is the square tile that does not cover a diagonal square.

Let t be the side length of the square tile covering the bottom-left unit square. Then every other square tile must cover a diagonal square, so by the same construction as in part (a), $n = 1+2+4+\cdots+2^{k-1}+t = 2^k+t-1$ for some positive integer k. Furthermore, the top $p(k) = 2^k - 1$ rows of the n-staircase must be tiled the same way as the minimal tiling of a $p(k)$-staircase. Therefore, the horizontal line between rows $p(k)$ and $p(k)+1$ does not pass through any square tiles. Let us call such a line a *fault line*. Similarly, the vertical line between columns t and $t+1$ is also a fault line. These two fault lines partition two $p(k)$-staircases.



If these two $p(k)$-staircases do not overlap, then $t = p(k)$, so $n = 2p(k)$. For example, the minimal tiling for $n = 2p(2) = 6$ is shown below.



Hence, assume that the two $p(k)$-staircases do overlap. The intersection of the two $p(k)$-staircases is a $[p(k)-t]$-staircase. Since this $[p(k)-t]$-staircase is tiled the same way as the top $p(k)-t$ rows of a minimal tiling of a $p(k)$-staircase, $p(k)-t = p(l)$ for some positive integer $l < k$, so $t = p(k)-p(l)$. Then

$$n = t + p(k) = 2p(k) - p(l).$$

Since $p(0) = 0$, we can summarize by saying that n must be of the form

$$n = 2p(k) - p(l) = 2^{k+1} - 2^l - 1,$$

where k is a positive integer and l is a nonnegative integer. Also, our argument shows how if n is of this form, then an n-staircase can be tiled with $n+1$ square tiles.

Finally, we observe that n is of this form if and only if the binary representation of n contains exactly one 0:

$$2^{k+1} - 2^l - 1 = \underbrace{11\ldots1}_{k-l \text{ 1s}}0\underbrace{11\ldots1}_{l \text{ 1s}}.$$

□

(2) Let A, B, P be three points on a circle. Prove that if a and b are the distances from P to the tangents at A and B and c is the distance from P to the chord AB, then $c^2 = ab$.

Solution. Let r be the radius of the circle, and let a' and b' be the respective lengths of PA and PB. Since $b' = 2r \sin\angle PAB = 2rc/a'$, $c = a'b'/(2r)$. Let AC be the diameter of the circle and H the foot of the perpendicular from P to AC. The similarity of the triangles ACP and APH imply that $AH : AP = AP : AC$ or $(a')^2 = 2ra$. Similarly, $(b')^2 = 2rb$. Hence

$$c^2 = \frac{(a')^2}{2r}\frac{(b')^2}{2r} = ab$$

as desired. □

Alternate Solution. Let E, F, G be the feet of the perpendiculars to the tangents at A and B and the chord AB, respectively. We need to show that $PE : PG = PG : GF$, where G is the foot of the perpendicular from P to AB. This suggest that we try to prove that the triangles EPG and GPF are similar.

Since PG is parallel to the bisector of the angle between the two tangents, $\angle EPG = \angle FPG$. Since $AEPG$ and $BFPG$ are concyclic quadrilaterals (having opposite angles right), $\angle PGE = \angle PAE$ and $\angle PFG = \angle PBG$. But $\angle PAE = \angle PBA = \angle PBG$, whence $\angle PGE = \angle PFG$. Therefore triangles EPG and GPF are similar.

The argument above with concyclic quadrilaterals only works when P lies on the shorter arc between A and B. The other case can be proved similarly. □

(3) Three speed skaters have a friendly race on a skating oval. They all start from the same point and skate in the same direction, but with different speeds that they maintain throughout the race. The slowest skater does 1 lap a minute, the fastest one does 3.14 laps a minute, and the middle one does L laps a minute for some $1 < L < 3.14$. The race ends at the moment when all three skaters again come together to the same point on the oval (which may differ from the starting

point.) Find how many different choices for L are there such that 117 passings occur before the end of the race. (A passing is defined when one skater passes another one. The beginning and the end of the race when all three skaters are at together are not counted as a passing.)

Solution. Assume that the length of the oval is one unit. Let $x(t)$ be the difference of distances that the slowest and the fastest skaters have skated by time t. Similarly, let $y(t)$ be the difference between the middle skater and the slowest skater. The path $(x(t), y(t))$ is a straight ray R in $\mathbb{R}^2$, starting from the origin, with slope depending on L. By assumption, $0 < y(t) < x(t)$.

One skater passes another one when either $x(t) \in \mathbb{Z}$, $y(t) \in \mathbb{Z}$ or $x(t) - y(t) \in \mathbb{Z}$. The race ends when both $x(t), y(t) \in \mathbb{Z}$.

Let $(a, b) \in \mathbb{Z}^2$ be the endpoint of the ray R. We need to find the number of such points satisfying:

(a) $0 < b < a$

(b) The ray R intersects $\mathbb{Z}^2$ at endpoints only.

(c) The ray R crosses 357 times the lines $x \in \mathbb{Z}$, $y \in \mathbb{Z}$, $y - x \in \mathbb{Z}$.

The second condition says that a and b are relatively prime. The ray R crosses $a-1$ of the lines $x \in \mathbb{Z}$, $b-1$ of the lines $y \in \mathbb{Z}$ and $a-b-1$ of the lines $x-y \in \mathbb{Z}$. Thus, we need $(a-1)+(b-1)+(a-b-1) = 117$, or equivalently, $2a-3 = 117$. That is $a = 60$.

Now b must be a positive integer less than and relatively prime to 60. The number of such b can be found using the Euler's ϕ function:

$$\phi(60) = \phi(2^2 \cdot 3 \cdot 5) = (2-1) \cdot 2 \cdot (3-1) \cdot (5-1) = 16.$$

Thus the answer is 16. □

Alternate Solution. First, let us name our skaters. From fastest to slowest, call them: A, B and C. (Abel, Bernoulli and Cayley?)

Now, it is helpful to consider the race from the viewpoint of C. Relative to C, both A and B complete a whole number of laps, since they both start and finish at C.

Let n be the number of laps completed by A relative to C, and let m be the number of laps completed by B relative to C. Note that: $n > m \in \mathbb{Z}^+$

Consider the number of minutes required to complete the race. Relative to C, A is moving with a speed of $3.14 - 1 = 2.14$ laps per minute and completes the race in $\frac{n}{2.14}$ minutes. Also relative to C, B is moving with a speed of $(L-1)$ laps per minute and completes the race in $\frac{m}{L-1}$ minutes. Since A and B finish the race together (when they both meet C):

$$\frac{n}{2.14} = \frac{m}{L-1} \qquad \Rightarrow \qquad L = 2.14\left(\frac{m}{n}\right) + 1.$$

Hence, there is a one-to-one relation between values of L and values of the postive proper fraction $\frac{m}{n}$. The fraction should be reduced, that is the pair (m, n) should be relatively prime, or else, with $k = \gcd(m, n)$, the race ends after n/k laps for A and m/k laps for B when they *first* meet C together.

It is also helpful to consider the race from the viewpoint of B. In this frame of reference, A completes only $n - m$ laps. Hence A *passes* B only $(n - m) - 1$ times, since the racers do not "pass" at the end of the race (nor at the beginning). Similarily A passes C only $n - 1$ times and B passes C only $m - 1$ times. The total number of passings is:

$$117 = (n-1) + (m-1) + (n-m-1) = 2n - 3 \quad \Rightarrow \quad n = 60$$

Hence the number of values of L equals the number of m for which the fraction $\frac{m}{60}$ is positive, proper and reduced. That is the number of positive integer values smaller than and relatively prime to 60. One could simply count: {1,7,11,13,17,...}, but Euler's ϕ function gives this number:

$$\phi(60) = \phi(2^2 \cdot 3 \cdot 5) = (2-1) \cdot 2 \cdot (3-1) \cdot (5-1) = 16.$$

Therefore, there are 16 values for L which give the desired number of passings.

Note that the actual values for the speeds of A and C do not affect the result. They could be any values, rational or irrational, just so long as they are different, and there will be 16 possible values for the speed of B between them. □

(4) Each vertex of a finite graph can be colored either black or white. Initially all vertices are black. We are allowed to pick a vertex P and change the color of P and all of its neighbours. Is it possible to change the colour of every vertex from black to white by a sequence of operations of this type?

Solution. The answer is yes. Proof by induction on the number n of vertices. If $n = 1$, this is obvious. For the induction assumption, suppose we can do this for any graph with $n-1$ vertices for some $n \geq 2$ and let X be a graph with n vertices which we will denote by $P_1, \ldots, P_{n+1}$.

Let us denote the "basic" operation of changing the color of P_i and all of its neighbours by f_i. Removing a vertex P_i from X (along with all edges connecting to P_i) and applying the induction assumption to the resulting smaller graph, we see that there exists a sequence of operations g_i (obtained by composing some f_j, with $j \neq i$) which changes the colour of every vertex in X, except for possibly P_i.

If g_i it also changes the color of P_i then we are done. So, we may assume that g_i does not change the colour of P for every $i = 1, \ldots, n$. Now consider two cases.

Case 1: n is even. Then composing $g_1, \ldots, g_n$ we will change the color of every vertex from white to black.

Case 2: n is odd. I claim that in this case X has a vertex with an even number of neighbours.

Indeed, denote the number of neighbours of P_i (or equivalently, the number of edges connected to P) by k_i. Then $P_1 + \cdots + P_{n+1} = 2e$, where e is the number of edges of X. Thus one of the numbers k_i has to be even as claimed.

After renumbering the vertices, we may assume that P_1 has $2k$ neighbours, say $P_2, \ldots, P_{2k+1}$. The composition of f_1 with $g_1, g_2, \ldots, g_{2k+1}$ will then change the colour of every vertex, as desired.

$\square$

(5) Let $P(x)$ and $Q(x)$ be polynomials with integer coefficients. Let $a_n = n! + n$. Show that if $P(a_n)/Q(a_n)$ is an integer for every n, then $P(n)/Q(n)$ is an integer for every integer n such that $Q(n) \neq 0$.

Solution. Imagine dividing $P(x)$ by $Q(x)$. We find that

$$\frac{P(x)}{Q(x)} = A(x) + \frac{R(x)}{Q(x)},$$

where $A(x)$ and $R(x)$ are polynomials with rational coefficients, and $R(x)$ is either identically 0 or has degree less than the degree of $Q(x)$.

By bringing the coefficients of $A(x)$ to their least common multiple, we can find a polynomial $B(x)$ with integer coefficients, and a positive integer b, such that $A(x) = B(x)/b$. Suppose first that $R(x)$ is not identically 0. Note that for any integer k, either $A(k) = 0$, or $|A(k)| \geq 1/b$. But whenever $|k|$ is large enough, $0 < |R(k)/Q(k)| < 1/b$, and therefore if n is large enough, $P(a_n)/Q(a_n)$ cannot be an integer.

So $R(x)$ is identically 0, and $P(x)/Q(x) = B(x)/b$ (at least whenever $Q(x) \neq 0$.)

Now let n be an integer. Then there are infinitely many integers k such that $n \equiv a_k \pmod{b}$. But $B(a_k)/b$ is an integer, or equivalently b divides $B(a_k)$. It follows that b divides $B(n)$, and therefore $P(n)/Q(n)$ is an integer. $\square$