Report of the Fortieth
Canadian Mathematical Olympiad
2008

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The Canadian Mathematical Olympiad (CMO) is an annual national mathematics competition sponsored by the Canadian Mathematical Society (CMS) and is administered by the Canadian Mathematical Olympiad Committee (CMO Committee), a sub-committee of the Mathematical Competitions Committee. The CMO was established in 1969 to provide an opportunity for students who performed well in various provincial mathematics competitions to compete at a national level. It also serves as preparation for those Canadian students competing at the International Mathematical Olympiad (IMO).

Students qualify to write the CMO by earning a sufficiently high score on the Canadian Open Mathematical Challenge (COMC). This year, the top 67 COMC scores were invited outright to write the CMO; all but seven accepted. Approximately 200 others, next in rank, were invited to send solutions to a Repêchage set of ten problems posted on line within a week to the University of Waterloo. Thirty-five students were invited on the basis of this, of which 31 accepted. I am grateful to Ian VanderBurgh for setting this up and assembling a team of markers, consisting of Ed Anderson, Lloyd Auckland, Ed Barbeau, Enzo Carli, Eddie Cheung, Rad de Peiza, Larry Rice, Jim Schurter, Ian VanderBurgh and Kyle Willick, to go through the 126 scripts received.

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I am very grateful to the CMO Committee members who submitted problems to be consdered for the 2008 competition: Ed Doolittle, Chris Fisher, Valeria Pandelieva, Naoki Sato, Adrian Tang and Jacob Tsimerman. The papers were marked by Ed Barbeau, Man-Duen Choi, Felix Recio and Lindsey Shorser. Thanks go to Tom Griffiths of London, ON and Kalle Karu of the University of British Columbia for validating the paper and to Joseph Khoury for translating the paper and the solutions into French. I am indebted for the hard work done at CMS headquarters by Susan Latreille and the Executive Director, Graham Wright, whose commitment and zeal is a vital ingredient of the success of the CMO.

Ed Barbeau, Chair
Canadian Mathematical Olympiad Committee

The 40th (2008) Canadian Mathematical Olympiad was written on Wednesday, March 26, 2008. A total of 96 students from 57 schools (52 in Canada, 3 in the US and 2 abroad) wrote the paper. Seven Canadian provinces were represented, with the number of contestants as follows:

BC (20)   AB (5)   SK (3)   MB (3)   ON (49)   QC (3)   NB (1).

The 2008 CMO consisted of five questions, each marked out of 7. The maximum score attained by a student was 28. The official contestants were grouped into four divisions according to their scores as follows:

Division	Range of Scores	No. of Students
I	21 - 28	8
II	13 - 19	17
III	7 - 12	29
IV	0 - 6	42

The following tables give the scores obtained on the COMC along with the corresponding scores obtained by candidates on the CMO. Students obtaining scores 72-80 qualified directly to write the CMO; students with scores 65-71 qualified through the Repêchage.

80 (26, 24, 15)
79 (21, 19, 15)
78 (23, 12, 10)
77 (28, 25, 18)
76 (12, 11, 11, 7, 6)
75 (15, 8, 5, 3)
74 (17, 17, 14, 14, 10, 6, 6, 3, 3, 3, 1)
73 (25, 21, 9, 9, 8, 7, 2, 2, 1, 0)
72 (15, 15, 11, 10, 9, 7, 5, 5, 4, 4, 2, 2, 1)

71 (19, 16, 11)
70 (4, 1)
69 (17, 4)
68 (19)
67 (13, 9, 8, 5, 1)
66 (11, 11, 10, 9, 9, 7, 6, 4, 1)
65 (10, 3, 3, 0)

FIRST PRIZE — Sun Life Financial Cup — $2000

Chen Sun
A.B. Lucas Secondary School, London, Ontario

SECOND PRIZE — $1500

Jonathan Schneider
University of Toronto Schools, Toronto, Ontario

THIRD PRIZE — $1000

Yan Li
Dr. Norman Bethune Collegiate Institute, Toronto, Ontario

HONOURABLE MENTIONS — $500

Dimitri Dziabenko
Don Mills Collegiate Institute
Toronto, ON

Neil Gurram
ICAE
Troy, MI

Danny Shi
Sir Winston Churchill High School
Calgary, AB

Jarno Sun
Western Canada High School
Calgary, AB

Tianyao Zhang
Sir John A. Macdonald Collegiate Institute
Toronto, ON

Division 1

21-28

Chen Sun	A.B. Lucas S.S.	ON
Jonathan Schneider	UTS	ON
Yan Li	Dr. Norman Bethune C.I.	ON
Dimitri Dziabenko	Don Mills C.I.	ON
Neil Gurram	ICAE	MI
Danny Shi	Sir Winston Churchill H.S.	AB
Jarno Sun	Western Canada H.S.	AB
Tianyao Zhang	Sir John A. Macdonald C.I.	ON

Division 2

13-19

Mohammad Babadi	Thornlea S.S.	ON
Frank Ban	Vincent Massey S.S.	ON
Yuhan Chen	Sir Winston Churchill C.V.I.	ON
Robin Cheng	Pinetree S.S.	BC
Bo Cheng Cui	West Vancouver S.S.	BC
Tony Feng	Phillips Academy	MA
David Field	Phillips Academy	MA
Yuting Huang	Johnston Heights S.S.	BC
Joe Kileel	Fredericton H.S.	NB
Zhiqiang Liu	Don Mills C.I.	ON
Jingyuan Mo	St. George's School	BC
Alexander Remorov	William Lyon Mackenzie King C.I.	ON
Jixuan Wang	Don Mills C.I.	ON
Weinan Peter Wen	Vincent Massey S.S.	ON
Anqi Zhang	Vincent Massey S.S.	ON
Linda Zhang	Western Canada H.S.	AB
Jonathan Zhou	Burnaby North S.S.	BC

Division 3

7-12

Golam Tahrif Bappi	Waterloo C.I.	ON
Shalev Ben David	Waterloo C.I.	ON
Ram Bhaskar	ICAE	MI
Philip Chen	Glenforest S.S.	ON
Weiliang Chen	Walter Murray C.I.	SK
Andrew Dhawan	The Woodlands School	ON
Henry Fung	Glenforest S.S.	ON
Fang Guo	Richmond Hill H.S.	ON
Tony Han	Jarvis C.I.	ON
Fan Jiang	Albert Campbell C.I.	ON
Heinrich Jiang	Vincent Massey S.S.	ON
Kwon Yong Jin	Phillips Academy	MA
Eric Gwangseung Kim	Prince of Wales S.S.	BC
Jung Hun Koh	Phillips Academcy	MA
Nikita Lvov	Marianopolis College	QC
Anupa Murali	Derryfield School	NH
Bill Pang	Sir Winston Churchill S.S.	BC
Owen Zhu Ren	Magee S.S.	BC
Mariya Sardarli	McKernan J.H.S.	AB
Alex Song	Waterloo C.I.	ON
Julian Sun	Sir Winston Churchill S.S.	BC
Ning Tang	London International Academy	ON
Ming Jing Wong	A.B. Lucas S.S.	ON
Xiao Xu	Georges Vanier S.S.	ON
Meng Ye	Marianopolis College	QC
Pei Jun Zhao	London Central S.S.	ON
Vincent Zhou	Dr. Norman Bethune C.I.	ON
Zimu Zhu	Richmond Hill H.S.	ON
Jonathan Zung	University of Toronto Schools	ON

Division 4

0-6

Shek Wah Chan	St. Paul's Co-ed. College	CN
Jerry Chen	Moscrop S.S.	BC
Lingjun Chen	Don Mills C.I.	ON
Chengcheng Gui	St. John's-Ravenscourt School	MB
Adam Halski	Kuwait English School	KW
Ding Henry Hao	Albert Campbell C.I.	ON
Matthew Harrisontraino	Marc Garneau C.I.	ON
Kevin He	Sir Winston Churchill S.S.	BC
Emily Wei-En Hsu	Branksome Hall School	ON
Yihuan Peter Huang	Kingston C.V.I.	ON
Navid Javadi	Earl Haig S.S.	ON
Chen Jiang	Central Technical School	ON
Yangzi Jiang	Waterloo C.I.	ON
Hee Woo Jun	Pinetree S.S.	BC
Randy Li	Phillips Academy	MA
Alex Liang	Dr. Norman Bethune C.I.	ON
William Lin	Albert Campbell C.I.	ON
Chieh Ming Liu	Fraser Heights S.S.	BC
Eric Liu	Sir Winston Churchill S.S.	BC
Xun Chao Max Liu	Port Moody S.S.	BC
David Ma	Marianopolis College	QC
Sudharshan Mohanram	ICAE	MI
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Susan Wang	Burnaby Central S.S.	BC
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Yunfan Zhang	Phillips Academy	MA
Dabo Zhao	White Oaks S.S.	ON
Steven Zhu	Sir Winston Churchill S.S.	BC
Yang Zhu	Albert Campbell C.I.	ON

40th Canadian Mathematical Olympiad

Wednesday, March 26, 2008

1. $ABCD$ is a convex quadrilateral for which AB is the longest side. Points M and N are located on sides AB and BC respectively, so that each of the segments AN and CM divides the quadrilateral into two parts of equal area. Prove that the segment MN bisects the diagonal BD.

2. Determine all functions f defined on the set of rational numbers that take rational values for which

$$f(2f(x) + f(y)) = 2x + y \ ,$$

for each x and y.

3. Let a, b, c be positive real numbers for which $a + b + c = 1$. Prove that

$$\frac{a - bc}{a + bc} + \frac{b - ca}{b + ca} + \frac{c - ab}{c + ab} \leq \frac{3}{2} \ .$$

4. Determine all functions f defined on the natural numbers that take values among the natural numbers for which

$$(f(n))^p \equiv n \mod f(p)$$

for all $n \in \mathbf{N}$ and all prime numbers p.

5. A *self-avoiding rook walk* on a chessboard (a rectangular grid of unit squares) is a path traced by a sequence of moves parallel to an edge of the board from one unit square to another, such that each begins where the previous move ended and such that no move ever crosses a square that has previously been crossed, *i.e.*, the rook's path is non-self-intersecting.

   Let $R(m, n)$ be the number of self-avoiding rook walks on an $m \times n$ (m rows, n columns) chessboard which begin at the lower-left corner and end at the upper-left corner. For example, $R(m, 1) = 1$ for all natural numbers m; $R(2, 2) = 2$; $R(3, 2) = 4$; $R(3, 3) = 11$. Find a formula for $R(3, n)$ for each natural number n.

40th Canadian Mathematical Olympiad

Wednesday, March 26, 2008

Solutions - CMO 2008

1. $ABCD$ is a convex quadrilateral in which AB is the longest side. Points M and N are located on sides AB and BC respectively, so that each of the segments AN and CM divides the quadrilateral into two parts of equal area. Prove that the segment MN bisects the diagonal BD.

Solution. Since $[MADC] = \frac{1}{2}[ABCD] = [NADC]$, it follows that $[ANC] = [AMC]$, so that $MN \| AC$. Let m be a line through D parallel to AC and MN and let BA produced meet m at P and BC produced meet m at Q. Then

$$[MPC] = [MAC] + [CAP] = [MAC] + [CAD] = [MADC] = [BMC]$$

whence $BM = MP$. Similarly $BN = NQ$, so that MN is a midline of triangle BPQ and must bisect BD.

2. Determine all functions f defined on the set of rationals that take rational values for which

$$f(2f(x) + f(y)) = 2x + y$$

for each x and y.

Solution 1. The only solutions are $f(x) = x$ for all rational x and $f(x) = -x$ for all rational x. Both of these readily check out.

Setting $y = x$ yields $f(3f(x)) = 3x$ for all rational x. Now replacing x by $3f(x)$, we find that

$$f(9x) = f(3f(3f(x)) = 3[3f(x)] = 9f(x) \ ,$$

for all rational x. Setting $x = 0$ yields $f(0) = 9f(0)$, whence $f(0) = 0$.

Setting $x = 0$ in the given functional equation yields $f(f(y)) = y$ for all rational y. Thus f is one-one onto. Applying f to the functional equation yields that

$$2f(x) + f(y) = f(2x + y)$$

for every rational pair (x, y).

Setting $y = 0$ in the functional equation yields $f(2f(x)) = 2x$, whence $2f(x) = f(2x)$. Therefore $f(2x) + f(y) = f(2x + y)$ for each rational pair (x, y), so that

$$f(u + v) = f(u) + f(v)$$

for each rational pair (u, v).

Since $0 = f(0) = f(-1) + f(1)$, $f(-1) = -f(1)$. By induction, it can be established that for each intger n and rational x, $f(nx) = nf(x)$. If $k = f(1)$, we can establish from this that $f(n) = nk$, $f(1/n) = k/n$ and $f(m/n) = mk/n$ for each integer pair (m, n). Thus $f(x) = kx$ for all rational x. Since $f(f(x)) = x$, we must have $k2 = 1$. Hence $f(x) = x$ or $f(x) = -x$. These check out.

Solution 2. In the functional equation, let

$$x = y = 2f(z) + f(w)$$

to obtain $f(x) = f(y) = 2z + w$ and

$$f(6z + 3w) = 6f(z) + 3f(w)$$

for all rational pairs (z, w). Set $(z, w) = (0, 0)$ to obtain $f(0) = 0$, $w = 0$ to obtain $f(6z) = 6f(z)$ and $z = 0$ to obtain $f(3w) = 3f(w)$ for all rationals z and w. Hence $f(6z + 3w) = f(6z) + f(3w)$. Replacing $(6z, 3w)$ by (u, v) yields

$$f(u + v) = f(u) + f(v)$$

for all rational pairs (u, v). Hence $f(x) = kx$ where $k = f(1)$ for all rational x. Substitution of this into the functional equation with $(x, y) = (1, 1)$ leads to $3 = f(3f(1)) = f(3k) = 3k2$, so that $k = \pm 1$. It can be checked that both $f(x) \equiv 1$ and $f(x) \equiv -1$ satisfy the equation.

Acknowledgment. The first solution is due to Man-Duen Choi and the second to Ed Doolittle.

3. Let a, b, c be positive real numbers for which $a + b + c = 1$. Prove that

$$\frac{a - bc}{a + bc} + \frac{b - ca}{b + ca} + \frac{c - ab}{c + ab} \le \frac{3}{2} \ .$$

Solution 1. Note that

$$1 - \frac{a - bc}{a + bc} = \frac{2bc}{1 - b - c + bc} = \frac{2bc}{(1 - b)(1 - c)} \ .$$

The inequality is equivalent to

$$\frac{2bc}{(1 - b)(1 - c)} + \frac{2ca}{(1 - c)(1 - a)} + \frac{2ab}{(1 - a)(1 - b)} \ge \frac{3}{2} \ .$$

Manipulation yields the equivalent

$$4(bc + ca + ab - 3abc) \ge 3(bc + ca + ab + 1 - a - b - c - abc) \ .$$

This simplifies to $ab + bc + ca \ge 9abc$ or

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} \ge 9 \ .$$

This is a consequence of the harmonic-arithmetic means inequality.

Solution 2. Observe that

$$a + bc = a(a + b + c) + bc = (a + b)(a + c)$$

and that $a + b = 1 - c$, with analogous relations for other permutations of the variables. Then

$$(b + c)(c + a)(a + b) = (1 - a)(1 - b)(1 - c) = (ab + bc + ca) - abc \ .$$

Putting the left side of the desired inequality over a common denominator, we find that it is equal to

$$\frac{(a-bc)(1-a)+(b-ac)(1-b)+(c-ab)(1-c)}{(b+c)(c+a)(a+b)} = \frac{(a+b+c)-(a2+b2+c2)-(bc+ca+ab)+3abc}{(b+c)(c+a)(a+b)}$$
$$= \frac{1-(a+b+c)2+(bc+ca+ab)+3abc}{(ab+bc+ca)-abc}$$
$$= \frac{(bc+ca+ab)+3abc}{(bc+bc+ab)-abc}$$
$$= 1+\frac{4abc}{(a+b)(b+c)(c+a)} \ .$$

Using the arithmetic-geometric means inequality, we obtain that

$$(a+b)(b+c)(c+a) = (a^2b+b^2c+c^2a)+(ab2+bc2+ca2)+2abc$$
$$\geq 3abc+3abc+2abc = 8abc \ ,$$

whence $4abc/[(a+b)(b+c)(c+a)] \leq \frac{1}{2}$. The desired result follows. Equality occurs exactly when $a = b = c = \frac{1}{3}$.

4. Find all functions f defined on the natural numbers that take values among the natural numbers for which

$$(f(n))^p \equiv n \quad \mod f(p)$$

for all $n \in \mathbf{N}$ and all prime numbers p.

Solution. The substitution $n = p$, a prime, yields $p \equiv (f(p))^p \equiv 0 \pmod{f(p)}$, so that p is divisible by $f(p)$. Hence, for each prime p, $f(p) = 1$ or $f(p) = p$.

Let $S = \{p : p \text{ is prime and } f(p) = p\}$. If S is infinite, then $f(n)^p \equiv n \pmod p$ for infinitely many primes p. By the little Fermat theorem, $n \equiv f(n)^p \equiv f(n)$, so that $f(n) - n$ is a multiple of p for infinitely many primes p. This can happen only if $f(n) = n$ for all values of n, and it can be verified that this is a solution.

If S is empty, then $f(p) = 1$ for all primes p, and any function satisfying this condition is a solution.

Now suppose that S is finite and non-empty. Let q be the largest prime in S. Suppose, if possible, that $q \geq 3$. Therefore, for any prime p exceeding q, $p \equiv 1 \pmod q$. However, this is not true. Let Q be the product of all the odd primes up to q. Then $Q + 2$ must have a prime factor exceeding q and at least one of them must be incongruent to 1 (mod q). (An alternative argument notes that Bertrand's postulate can turn up a prime p between q and $2q$ which fails to satisfy $p \equiv 1 \mod q$.)

The only remaining case is that $S = \{2\}$. Then $f(2) = 2$ and $f(p) = 1$ for every odd prime p. Since $f(n)2 \equiv n \pmod 2$, $f(n)$ and n must have the same parity. Conversely, any function f for which $f(n) \equiv n \pmod 2$ for all n, $f(2) = 2$ and $f(p) = 1$ for all odd primes p satisfies the condition.

Therefore the only solutions are
- $f(n) = n$ for all $n \in \mathbf{N}$;
- any function f with $f(p) = 1$ for all primes p;
- any function for which $f(2) = 2$, $f(p) = 1$ for primes p exceeding 2 and $f(n)$ and n have the same parity.

5. A *self-avoiding rook walk* on a chessboard (a rectangular grid of squares) is a path traced by a sequence of rook moves parallel to an edge of the board from one unit square to another, such that each begins where the previous move ended and such that no move ever crosses a square that has previously been crossed, *i.e.*, the rook's path is non-self-intersecting.

   Let $R(m, n)$ be the number of self-avoiding rook walks on an $m \times n$ (m rows, n columns) chessboard which begin at the lower-left corner and end at the upper-left corner. For example, $R(m, 1) = 1$ for all natural numbers m; $R(2,2) = 2$; $R(3,2) = 4$; $R(3,3) = 11$. Find a formula for $R(3,n)$ for each natural number n.

Solution 1. Let $r_n = R(3,n)$. It can be checked directly that $r_1 = 1$ and $r_2 = 4$. Let $1 \le i \le 3$ and $1 \le j$; let (i,j) denote the cell in the ith row from the bottom and the jth column from the left, so that the paths in question go from $(1,1)$ to $(3,1)$.

Suppose that $n \ge 3$. The rook walks fall into exactly one of the following six categories:

(1) One walk given by $(1,1) \to (2,1) \to (3,1)$.

(2) Walks that avoid the cell $(2,1)$: Any such walk must start with $(1,1) \to (1,2)$ and finish with $(3,2) \to (3,1)$; there are r_{n-1} such walks.

(3) Walks that begin with $(1,1) \to (2,1) \to (2,2)$ and never return to the first row: Such walks enter the third row from $(2,k)$ for some k with $2 \le k \le n$ and then go along the third row leftwards to $(3,1)$; there are $n-1$ such walks.

(4) Walks that begin with $(1,1) \to (2,1) \to \cdots \to (2,k) \to (1,k) \to (1,k+1)$ and end with $(3,k+1) \to (3,k) \to (3,k-1) \to \cdots \to (3,2) \to (3,1)$ for some k with $2 \le k \le n-1$; there are $r_{n-2} + r_{n-3} + \cdots + r_1$ such walks.

(5) Walks that are the horizontal reflected images of walks in (3) that begin with $(1,1) \to (2,1)$ and never enter the third row until the final cell; there are $n-1$ such walks.

(6) Walks that are horizontal reflected images of walks in (5); there are $r_{n-2} + r_{n-3} + \cdots + r_1$ such walks.

Thus, $r_3 = 1 + r_2 + 2(2 + r_1) = 11$ and, for $n \ge 3$,

$$\begin{aligned} r_n &= 1 + r_{n-1} + 2[(n-1) + r_{n-2} + r_{n-3} + \cdots + r_1] \\ &= 2n - 1 + r_{n-1} + 2(r_{n-2} + \cdots + r_1) \ , \end{aligned}$$

and

$$r_{n+1} = 2n + 1 + r_n + 2(r_{n-1} + r_{n-2} + \cdots + r_1) \ .$$

Therefore

$$r_{n+1} - r_n = 2 + r_n + r_{n-1} \Longrightarrow r_{n+1} = 2 + 2r_n + r_{n-1} \ .$$

Thus

$$r_{n+1} + 1 = 2(r_n + 1) + (r_{n-1} + 1) \ ,$$

whence

$$r_n + 1 = \frac{1}{2\sqrt{2}}(1+\sqrt{2})^{n+1} - \frac{1}{2\sqrt{2}}(1-\sqrt{2})^{n+1} \ ,$$

and

$$r_n = \frac{1}{2\sqrt{2}}(1+\sqrt{2})^{n+1} - \frac{1}{2\sqrt{2}}(1-\sqrt{2})^{n+1} - 1 \ .$$

Solution 2. Employ the same notation as in Solution 1. We have that $r_1 = 1$, $r_2 = 4$ and $r_3 = 11$. Let $n \ge 3$. Consider the situation that there are r_{n+1} columns. There are basically three types of rook walks.

Type 1. There are four rook walks that enter only the first two columns.

Type 2. There are $3r_{n-1}$ rooks walks that do not pass between the second and third columns in the middle row (in either direction), *viz.* r_{n-1} of each of the types:

$$(1,1) \longrightarrow (1,2) \longrightarrow (1,3) \longrightarrow \cdots \longrightarrow (3,3) \longrightarrow (3,2) \longrightarrow (3,1) \ ;$$

$$(1,1) \longrightarrow (2,1) \longrightarrow (2,2) \longrightarrow (1,2) \longrightarrow (1,3) \longrightarrow \cdots \longrightarrow (3,3) \longrightarrow (3,2) \longrightarrow (3,1) \ ;$$

$$(1,1) \longrightarrow (1,2) \longrightarrow (1,3) \longrightarrow \cdots \longrightarrow (3,3) \longrightarrow (3,2) \longrightarrow (2,2) \longrightarrow (2,1) \longrightarrow (3,1) \ .$$

Type 3. Consider the rook walks that pass between the second and third column along the middle row.

They are of Type 3a:

$$(1,1) \longrightarrow * \longrightarrow (2,2) \longrightarrow (2,3) \longrightarrow \cdots \longrightarrow (3,3) \longrightarrow (3,2) \longrightarrow (3,1) \ ,$$

or Type 3b:

$$(1,1) \longrightarrow (1,2) \longrightarrow (1,3) \longrightarrow \cdots \longrightarrow (2,3) \longrightarrow (2,2) \longrightarrow * \longrightarrow (3,1) \ ,$$

where in each case the asterisk stands for one of two possible options.

We can associate in a two-one way the walks of Type 3a to a rook walk on the last n columns, namely

$$(1,2) \longrightarrow (2,2) \longrightarrow (2,3) \longrightarrow \cdots \longrightarrow (3,3) \longrightarrow (3,2)$$

and the walks of Type 3b to a rook walk on the last n columns, namely

$$(1,2) \longrightarrow (1,3) \longrightarrow \cdots \longrightarrow (2,3) \longrightarrow (2,2) \longrightarrow (3,2) \ .$$

The number of rook walks of the latter two types together is $r_n - 1 - r_{n-1}$. From the number of rook walks on the last n columns, we subtract one for $(1,2) \to (2,2) \to (3,2)$ and r_{n-1} for those of the type

$$(1,2) \longrightarrow (1,3) \longrightarrow \cdots \longrightarrow (3,3) \longrightarrow (2,3) \ .$$

Therefore, the number of rook walks of Type 3 is $2(r_n - 1 - r_{n-1})$ and we find that

$$r_{n+1} = 4 + 3r_{n-1} + 2(r_n - 1 - r_{n-1}) = 2 + 2r_n + r_{n-1} \ .$$

We can now complete the solution as in Solution 1.

Solution 3. Let $S(3,n)$ be the set of self-avoiding rook walks in which the rook occupies column n but does not occupy column $n+1$. Then $R(3,n) = |S(3,1)| + |S(3,2)| + \cdots + |S(3,n)|$. Furthermore, topological considerations allow us to break $S(3,n)$ into three disjoint subsets $S_1(3,n)$, the set of paths in which corner $(1,n)$ is not occupied, but there is a path segment $(2,n) \longrightarrow (3,n)$; $S_2(3,n)$, the set of paths in which corners $(1,n)$ and $(3,n)$ are both occupied by a path $(1,n) \longrightarrow (2,n) \longrightarrow (3,n)$; and $S_3(3,n)$, the set of paths in which corner $(3,n)$ is not occupied but there is a path segment $(1,n) \longrightarrow (2,n)$. Let $s_i(n) = |S_i(3,n)|$ for $i = 1,2,3$. Note that $s_1(1) = 0$, $s_2(1) = 1$ and $s_3(1) = 0$. By symmetry, $s_1(n) = s_3(n)$ for every positive n. Furthermore, we can construct paths in $S(3,n+1)$ by "bulging" paths in $S(3,n)$, from which we obtain

$$\begin{aligned} s_1(n+1) &= s_1(n) + s_2(n) \ ; \\ s_2(n+1) &= s_1(n) + s_2(n) + s_3(n) \ ; \\ s_3(n+1) &= s_2(n) + s_3(n) \ ; \end{aligned}$$

or, upon simplification,

$$\begin{aligned} s_1(n+1) &= s_1(n) + s_2(n) \ ; \\ s_2(n+1) &= 2s_1(n) + s_2(n) \ . \end{aligned}$$

Hence, for $n \geq 2$,

$$\begin{aligned} s_1(n+1) &= s_1(n) + 2s_1(n-1) + s_2(n-1) \\ &= s_1(n) + 2s_1(n-1) + s_1(n) - s_1(n-1) \\ &= 2s_1(n) + s_1(n-1) \ . \end{aligned}$$

and

$$\begin{aligned} s_2(n+1) = 2s_1(n) + s_2(n) &= 2s_1(n-1) + 2s_2(n-1) + s_2(n) \\ &= s_2(n) - s_2(n-1) + 2s_2(n-1) + s_2(n) \\ &= 2s_2(n) + s_2(n-1) \ . \end{aligned}$$

We find that

$$s_1(n) = \frac{1}{2\sqrt{2}}(1+\sqrt{2})^{n-1} - \frac{1}{2\sqrt{2}}(1-\sqrt{2})^{n-1} ;$$
$$s_2(n) = \frac{1}{2}(1+\sqrt{2})^{n-1} + \frac{1}{2}(1-\sqrt{2})^{n-1} .$$

Summing a geometric series yields that

$$\begin{aligned} R(3,n) &= (s_2(1)+\cdots+s_2(n)) + 2(s_1(1)+\cdots+s_1(n)) \\ &= \left(\frac{1}{2}+\frac{1}{\sqrt{2}}\right)\left(\frac{(1+\sqrt{2})^n-1}{\sqrt{2}}\right) + \left(\frac{1}{2}-\frac{1}{\sqrt{2}}\right)\left(\frac{(1-\sqrt{2})^n-1}{-\sqrt{2}}\right) \\ &= \left(\frac{1}{2\sqrt{2}}\right)[(1+\sqrt{2})^{n+1}-(1-\sqrt{2})^{n+1}] - 1 . \end{aligned}$$

The formula agrees with $R(3,1)=1$, $R(3,2)=4$ and $R(3,3)=11$.

Acknowledgment. The first two solutions are due to Man-Duen Choi, and the third to Ed Doolittle.

THE GRADERS' REPORT

The grading was done by Ed Barbeau, Man-Duen Choi, Felix Recio and Lindsey Shorser. The papers in the top three divisions and most of those in the fourth were marked independently by two markers and all papers in division I were reviewed by four markers, as well as many in division II.

The paper was more difficult this year than in 2007, and it was decided to have but one geometry problem, but that one reasonably easy. There was less opportunity for part marks, and the final results reflect this.

As the papers were being marked, it became evident that there is an increasing gap between the background knowledge and sophistication of the top ten or twenty students and the rest, a matter that should be of some concern to those concerned about an adequate supply of prepared students entering university engineering and science programs. This makes it more difficult to prepare a paper, even for sixty or so invited students, that is a challenge for the top students while being accessible to the rest.

The contest buffs among the students now know as a matter of course results that the reader of this report might not have encountered until their undergraduate or even graduate years, if at all. They know how to solve linear recursion, are familiar with more advanced results of Euclidean geometry such as the theorems of Ceva and Menelaus, or the Ptolemy inequality, are adept at using transformation arguments (including inversion in a circle) in plane geometry, have at their command basic results in graph theory, are familiar with modular arithmetic and number theory results such as Fermat's Little Theorem and the Chinese Remainder Theorem, know some differential calculus and have a larger supply of inequalities in their quivers, beyond the arithmetic-geometric means and Cauchy-Schwarz inequalities; two students turned to the Muirhead majorization inequalities to solve Question 3. The problems committee did not have most of these advanced techniques in mind in setting the problems; they were designed to be solved through basic reasoning and competent application of standard techniques.

Many students instinctively turn to formulas and computation without taking the time to consider the essence of a problem situation and try an argument that is more attuned to its structure. This was evident in Questions 1, 2 and 4.

Students invited to write the Canadian Mathematical Olympiad cannot expect to do well without preparing for the examination. This should involve reviewing problems of past contests of this level, as well as strengthening their backgrounds in areas in which the school curriculum is weak - geometry, basic combinatorics, inequalities and polynomial algebra. Plenty of material to this end can be obtained through the Canadian Mathematical Society (www.cms.math.ca) and the Mathematical Association of America (www.maa.org).

One flaw in the standard school syllabus became evident in the use either advanced or “clunky” techniques in solving problems. Some students are exposed superficially to more advanced mathematics such as trigonometry and calculus before they have become familiar and facile with more elementary geometry and algebra.

Each problem was marked out of 7.
The marks awarded on the several problems are given in the following table:

Marks	#1	#2	#3	#4	#5
7	30	7	23	0	1
6	1	3	2	0	0
5	4	1	3	1	0
4	6	4	2	2	1
3	6	9	8	4	1
2	8	18	14	9	3
1	15	29	15	29	15
0	13	15	25	12	33
-	13	10	4	39	42

Problem 1: The solution mainly relied on the fact that two triangles on the same base and between the same parallels have equal area. Many students, however, cluttered their solutions by inserting altitudes and using the half base-times-height formula; while they usually got it correct, the result was a solution twice as long as it should have been. Several students resorted to analytic geometry, a method particularly cumbersome when the up-and-down formula for the area of a triangle was employed.

Problem 2: Functional equations problems tend to become harder when unwarranted restrictions are imposed on the function, and this situation was no exception. Some students, having had some calculus, reached immediately for the derivative, and got bogged down. They apparently do not realize that not every function admits a derivative. Quite a few students implicitly assumed continuity and drew from the bijective character of f that it was monotone. In another direction, certain candidates took f to be a polynomial, in some cases a linear polynomial, and worked from there. Some solvers were familiar with the result that a function satisfying the equation $g(x + y) = g(x) + g(y)$ on the rationals had the form $g(x) = kx$, and the statement of this fact without a proof was accepted by the markers.

Problem 3: Performance on this problem was better than expected, and several effective methods were adduced to handling it. While an astute application of the arithmetic-goemetric means or harmonic-arithmetic means inequality sufficed to bring it home, a few candidates brought much heavier machinery into play without making the solution any more efficient.

Problem 4: This was a tricky problem, but not done as well as expected. The possibility of congruence modulo 1 did not cause any difficulty except for one or two cases. Many candidates were familiar with the little Fermat theorem (and this should be regarded as an essential part of the preparation for the CMO); nevertheless, fewer identified the solutions $f(n) = n$ or $f(p) = 1$ for all primes p than we had hoped. Most of the candidates who gained points on this problem did derive early on that $f(p)$ divided p and so had to be either 1 or p, for each prime p. A few thought that one of these alternatives always had to occur. Quite a few set $f(n) = n + kp$, at which point the solutions generally became hopelessly complicated.

Problem 5: This was intended to be a hard problem, and it did require the students to solve a linear recursion. However, four marks were available to any student who could set up the recursion; the main burden of the problem was the analysis of the different types of rook walks and the formulation of an induction process. We were disappointed at how few students were able to reason carefully enough to get even a good start at this. A couple of candidates misread the problem and had the walk ending in the upper-right corner.