Report of the Thirty Seventh Canadian Mathematical Olympiad 2005

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The Canadian Mathematical Olympiad (CMO) is an annual national mathematics competition sponsored by the Candian Mathematical Society (CMS) and is administered by the Canadian Mathematical Olympiad Committee (CMO Committee), a sub-committee of the Mathematical Competitions Committee. The CMO was established in 1969 to provide an opportunity for students who performed well in various provincial mathematics competitions to compete at a national level. It also serves as preparation for those Canadian students competing at the International Mathematical Olympiad (IMO).

Students qualify to write the CMO by earning a sufficiently high score on the Canadian Open Mathematical Challenge (COMC). Students may also be nominated to write the CMO by a provincial coordinator.

The Society is grateful for support from the Sun Life Assurance Company of Canada as the Major Sponsor of the 2005 Canadian Mathematical Olympiad and the other sponsors which include: the Ministry of Education of Ontario; the Ministry of Education of Quebec; Alberta Learning; the Department of Education, New Brunswick; the Department of Education, Newfoundland and Labrador; the Department of Education, the Northwest Territories; the Department of Education of Saskatchewan; the Department of Mathematics and Statistics, University of Winnipeg; the Department of Mathematics and Statistics, University of New Brunswick at Fredericton; the Centre for Education in Mathematics and Computing, University of Waterloo; the Department of Mathematics and Statistics, University of Ottawa; the Department of Mathematics, University of Toronto; the Department of Mathematics, University of British Columbia; Nelson Thompson Learning; John Wiley and Sons Canada Ltd.; A.K. Peters and Maplesoft.

The provincial coordinators of the CMO are Peter Crippin, University of Waterloo ON; John Denton, Dawson College QC; Diane Dowling, University of Manitoba; Harvey Gerber, Simon Fraser University BC; Gareth J. Griffith, University of Saskatchewan; Jacques Labelle, Université du Québec à Montréal; Peter Minev, University of Alberta; Gordon MacDonald, University of Prince Edward Island; Roman Mureika, University of New Brunswick; Thérèse Ouellet, Université de Montréal QC; Donald Rideout, Memorial University of Newfoundland.

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Terry Visentin, Chair
Canadian Mathematical Olympiad Committee

The 37th (2005) Canadian Mathematical Olympiad was held on Wednesday, March 30th, 2005. A total of 75 students from 48 schools in eight Canadian provinces wrote the paper. One Canadian student wrote the exam in Singapore. The number of contestants from each province was as follows:

BC(8) AB(10) SK(1) MB(3) ON(47) QC(3) NB(1) PEI(1)

The 2005 CMO consisted of five questions. Each question was worth 7 marks for a total maximum score of m=35. The contestants' performances were grouped into four divisions as follows.

Division	Range of Scores	No. of Students
I	$24 \leq m < 35$	10
II	$18 \leq m < 24$	15
III	$14 \leq m < 18$	19
IV	$0 \leq m < 14$	31

FIRST PRIZE — Sun Life Financial Cup — $2000
Peng Shi
Sir John A. MacDonald Collegiate Institute, Agincourt, Ontario

SECOND PRIZE — $1500
Richard Peng
Vaughan Road Academy, Toronto, Ontario

THIRD PRIZE — $1000
Yufei Zhao
Don Mills Collegiate Institute

HONOURABLE MENTIONS — $500
Boris Braverman
Sir Winston Churchill High School, Calgary, Alberta

Elyot Grant
Cameron Heights Collegiate Institute, Kitchener, Ontario

Zheng Guo
Western Canada High School, Calgary, Alberta

Oleg Ivrii
Don Mills Collegiate Institute,
Don Mills, Ontario

Lin Fei
Don Mills Collegiate Institute,
Don Mills, Ontario

Dong Uk (David) Rhee
McNally School, Edmonton, Alberta

Shaun White
Vincent Massey Secondary School,
Windsor, Ontario

Division 2
$18 \leq m < 24$

Farzin Barekat	Sutherland Secondary School	BC
Rongtao Dan	Point Grey Secondary School	BC
Bo Hong Deng	Jarvis Collegiate Institute	ON
William Fu	A.Y. Jackson Secondary School	ON
Kent Huynh	University of Toronto Schools	ON
Aidin Kashigar	Sir Frederick Banting Secondary School	ON
Viktoriya Krakovna	Vaughan Road Academy	ON
William Ma	Waterloo Collegiate Institute	ON
Jennifer Park	Bluevale Collegiate Institute	ON
Karol Przybytkowski	Marianopolis College	QC
Luke Schaeffer	Centennial C. & V.I.	ON
Geoffrey Siu	London Central Secondary School	ON
Alex Wice	Leaside High School	ON
Brian Yu	Old Scona Academic High School	AB
Allen Zhang	St. George's School	BC

Division 3
$14 \leq m < 18$

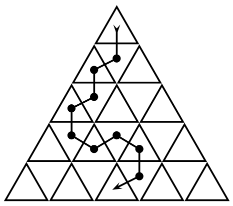
Eunse Chang	Don Mills Collegiate Institute	ON
Yiru Chen	Semiahmoo Secondary School	BC
Francis Chung	A.B. Lucas Secondary School	ON
Shawn C. Eastwood	Canadian International School (Singapoor)	CA
Weixi Fan	Dover Bay Secondary School	BC
Mostafa Fatehi	Colonel Gray Senior High School	PEI
Yingfen Huang	The Woodlands School	ON
Kevin Lam	St. John's-Ravenscourt School	MB
Taotao Liu	Vincent Massey Secondary School	ON
Nick Murdoch	London Central Secondary School	ON
Chuanming Qi	Jarvis Collegiate Institute	ON
Roman Shapiro	Vincent Massey Secondary School	ON
Jimmy Shen	Vincent Massey Secondary School	ON
Sarah Sun	Holy Trinity Academy	AB
Ruiqing Wang	Vanier College	QC
Malka Wrigley	Old Scona Academic High School	AB
Wenxin Xu	Don Mills Collegiate Institute	ON
Qi Yao	Glenforest Secondary School	ON
Vivian Zhang	Bayview Secondary School	ON

Division 4
$0 \leq m < 14$

Larry Chang	Seaquam Secondary School	BC
Harry Chang	A.B. Lucas Secondary School	ON
Chan Ching Chen	St. George's School	BC
Dmitri Dziabenko	Don Mills Collegiate Institute	ON
Dong (Polly) Han	Western Canada High School	AB
Ari Jeon	North Toronto Collegiate Institute	ON
Sha Jin	York Mills Collegiate Institute	ON
Ying Li	Lisgar Collegiate Institute	ON
Chen Li	Fredericton High School	NB
Ye Qing Lin	Earl Of March Secondary School	ON
Elliot Lipnowski	St. John's-Ravenscourt School	MB
Shengyan Liu	Martingrove Collegiate Institute	ON
Yuchen Mu	St. John's-Ravenscourt School	MB
Yongho Park	Richmond Hill High School	ON
Alex Qi	Waterloo Collegiate Institute	ON
Difu Shi	Glebe Collegiate Institute	ON
Hunter Song	A.Y. Jackson Secondary School	ON
Chen Sun	Tom Griffiths Home School	ON
Jia Xi Sun	Walter Murray Collegiate Institute	SK
Eric Tran	Western Canada High School	AB
Kuan Chieh Tseng	Yale Secondary School	BC
Jenny Wang	Don Mills Collegiate Institute	ON
David Wang	London Central Secondary School	ON
Frederic Weigand Warr	College Jean-De-Brebeuf	QC
Steven Wu	A.Y. Jackson Secondary School	ON
Xiaodi Wu	University of Toronto Schools	ON
Rui Xue	Martingrove Collegiate Institute	ON
Yiyi Yang	Western Canada High School	AB
Johnny Zhang	William Lyon Mackenzie C.I.	ON
Ken Zhang	Western Canada High School	AB
Ryan Zhou	Adam Scott Collegiate Vocational Institute	ON

37th Canadian Mathematical Olympiad
March 30, 2005

1. Consider an equilateral triangle of side length n, which is divided into unit triangles, as shown. Let $f(n)$ be the number of paths from the triangle in the top row to the middle triangle in the bottom row, such that adjacent triangles in our path share a common edge and the path never travels up (from a lower row to a higher row) or revisits a triangle. An example of one such path is illustrated below for $n = 5$. Determine the value of $f(2005)$.



2. Let (a, b, c) be a Pythagorean triple, *i.e.*, a triplet of positive integers with $a^2 + b^2 = c^2$.

   a) Prove that $(c/a + c/b)^2 > 8$.

   b) Prove that there does not exist any integer n for which we can find a Pythagorean triple (a, b, c) satisfying $(c/a + c/b)^2 = n$.

3. Let S be a set of $n \geq 3$ points in the interior of a circle.

   a) Show that there are three distinct points $a, b, c \in S$ and three distinct points A, B, C on the circle such that a is (strictly) closer to A than any other point in S, b is closer to B than any other point in S and c is closer to C than any other point in S.

   b) Show that for no value of n can four such points in S (and corresponding points on the circle) be guaranteed.

4. Let ABC be a triangle with circumradius R, perimeter P and area K. Determine the maximum value of KP/R^3.

5. Let's say that an ordered triple of positive integers (a, b, c) is *n-powerful* if $a \leq b \leq c$, $\gcd(a, b, c) = 1$, and $a^n + b^n + c^n$ is divisible by $a + b + c$. For example, $(1, 2, 2)$ is 5-powerful.

   a) Determine all ordered triples (if any) which are n-powerful for all $n \geq 1$.

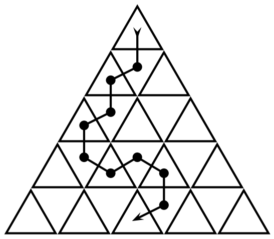
   b) Determine all ordered triples (if any) which are 2004-powerful and 2005-powerful, but not 2007-powerful.

   [Note that $\gcd(a, b, c)$ is the greatest common divisor of a, b and c.]

Solutions to the 2005 CMO

written March 30, 2005

1. Consider an equilateral triangle of side length n, which is divided into unit triangles, as shown. Let $f(n)$ be the number of paths from the triangle in the top row to the middle triangle in the bottom row, such that adjacent triangles in our path share a common edge and the path never travels up (from a lower row to a higher row) or revisits a triangle. An example of one such path is illustrated below for $n = 5$. Determine the value of $f(2005)$.

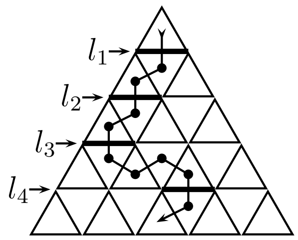


Solution

We shall show that $f(n) = (n-1)!$.

Label the horizontal line segments in the triangle l_1, l_2, ... as in the diagram below. Since the path goes from the top triangle to a triangle in the bottom row and never travels up, the path must cross each of $l_1, l_2, \ldots, l_{n-1}$ exactly once. The diagonal lines in the triangle divide l_k into k unit line segments and the path must cross exactly one of these k segments for each k. (In the diagram below, these line segments have been highlighted.) The path is completely determined by the set of $n-1$ line segments which are crossed. So as the path moves from the kth row to the $(k+1)$st row, there are k possible line segments where the path could cross l_k. Since there are $1 \cdot 2 \cdot 3 \cdots (n-1) = (n-1)!$ ways that the path could cross the $n-1$ horizontal lines, and each one corresponds to a unique path, we get $f(n) = (n-1)!$.

Therefore $f(2005) = (2004)!$.



2. Let (a, b, c) be a Pythagorean triple, *i.e.*, a triplet of positive integers with $a^2+b^2=c^2$.

   a) Prove that $(c/a + c/b)^2 > 8$.

   b) Prove that there does not exist any integer n for which we can find a Pythagorean triple (a, b, c) satisfying $(c/a + c/b)^2 = n$.

a) **Solution 1**

Let (a, b, c) be a Pythagorean triple. View a, b as lengths of the legs of a right angled triangle with hypotenuse of length c; let θ be the angle determined by the sides with lengths a and c. Then

$$\begin{aligned}\left(\frac{c}{a}+\frac{c}{b}\right)^2 &= \left(\frac{1}{\cos\theta}+\frac{1}{\sin\theta}\right)^2 = \frac{\sin^2\theta+\cos^2\theta+2\sin\theta\cos\theta}{(\sin\theta\cos\theta)^2}\\ &= 4\left(\frac{1+\sin 2\theta}{\sin^2 2\theta}\right) = \frac{4}{\sin^2 2\theta}+\frac{4}{\sin 2\theta}\end{aligned}$$

Note that because $0 < \theta < 90^\circ$, we have $0 < \sin 2\theta \le 1$, with equality only if $\theta = 45^\circ$. But then $a = b$ and we obtain $\sqrt{2} = c/a$, contradicting a, c both being integers. Thus, $0 < \sin 2\theta < 1$ which gives $(c/a + c/b)^2 > 8$.

Solution 2

Defining θ as in Solution 1, we have $c/a + c/b = \sec\theta + \csc\theta$. By the AM-GM inequality, we have $(\sec\theta + \csc\theta)/2 \ge \sqrt{\sec\theta\csc\theta}$. So

$$c/a + c/b \ge \frac{2}{\sqrt{\sin\theta\cos\theta}} = \frac{2\sqrt{2}}{\sqrt{\sin 2\theta}} \ge 2\sqrt{2}.$$

Since a, b, c are integers, we have $c/a + c/b > 2\sqrt{2}$ which gives $(c/a + c/b)^2 > 8$.

Solution 3

By simplifying and using the AM-GM inequality,

$$\left(\frac{c}{a}+\frac{c}{b}\right)^2 = c^2\left(\frac{a+b}{ab}\right)^2 = \frac{(a^2+b^2)(a+b)^2}{a^2b^2} \ge \frac{2\sqrt{a^2b^2}\,(2\sqrt{ab})^2}{a^2b^2} = 8,$$

with equality only if $a = b$. By using the same argument as in Solution 1, a cannot equal b and the inequality is strict.

Solution 4

$$\begin{aligned}\left(\frac{c}{a}+\frac{c}{b}\right)^2 &= \frac{c^2}{a^2}+\frac{c^2}{b^2}+\frac{2c^2}{ab} = 1+\frac{b^2}{a^2}+\frac{a^2}{b^2}+1+\frac{2(a^2+b^2)}{ab}\\ &= 2+\left(\frac{a}{b}-\frac{b}{a}\right)^2+2+\frac{2}{ab}\Big((a-b)^2+2ab\Big)\\ &= 4+\left(\frac{a}{b}-\frac{b}{a}\right)^2+\frac{2(a-b)^2}{ab}+4 \ge 8,\end{aligned}$$

with equality only if $a = b$, which (as argued previously) cannot occur.

b) **Solution 1**

Since $c/a + c/b$ is rational, $(c/a + c/b)^2$ can only be an integer if $c/a + c/b$ is an integer. Suppose $c/a + c/b = m$. We may assume that $\gcd(a, b) = 1$. (If not, divide the common factor from (a, b, c), leaving m unchanged.)

Since $c(a+b) = mab$ and $\gcd(a, a+b) = 1$, a must divide c, say $c = ak$. This gives $a^2 + b^2 = a^2k^2$ which implies $b^2 = (k^2 - 1)a^2$. But then a divides b contradicting the fact that $\gcd(a, b) = 1$. Therefore $(c/a + c/b)^2$ is not equal to any integer n.

Solution 2

We begin as in Solution 1, supposing that $c/a + c/b = m$ with $\gcd(a, b) = 1$. Hence a and b are not both even. It is also the case that a and b are not both odd, for then $c^2 = a^2 + b^2 \equiv 2 \pmod 4$, and perfect squares are congruent to either 0 or 1 modulo 4. So one of a, b is odd and the other is even. Therefore c must be odd.

Now $c/a + c/b = m$ implies $c(a+b) = mab$, which cannot be true because $c(a+b)$ is odd and mab is even.

3. Let S be a set of $n \geq 3$ points in the interior of a circle.

   a) Show that there are three distinct points $a, b, c \in S$ and three distinct points A, B, C on the circle such that a is (strictly) closer to A than any other point in S, b is closer to B than any other point in S and c is closer to C than any other point in S.

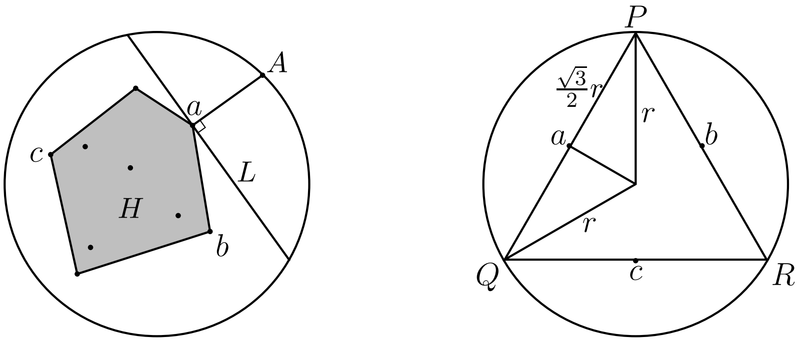
   b) Show that for no value of n can four such points in S (and corresponding points on the circle) be guaranteed.

Solution 1

a) Let H be the smallest convex set of points in the plane which contains S.† Take 3 points $a, b, c \in S$ which lie on the boundary of H. (There must always be at least 3 (but not necessarily 4) such points.)

Since a lies on the boundary of the convex region H, we can construct a chord L such that no two points of H lie on opposite sides of L. Of the two points where the perpendicular to L at a meets the circle, choose one which is on a side of L not containing any points of H and call this point A. Certainly A is closer to a than to any other point on L or on the other side of L. Hence A is closer to a than to any other point of S. We can find the required points B and C in an analogous way and the proof is complete.

[Note that this argument still holds if all the points of S lie on a line.]



(a) (b)

b) Let PQR be an equilateral triangle inscribed in the circle and let a, b, c be mid-points of the three sides of $\triangle PQR$. If r is the radius of the circle, then every point on the circle is within $(\sqrt{3}/2)r$ of one of a, b or c. (See figure (b) above.) Now $\sqrt{3}/2 < 9/10$, so if S consists of a, b, c and a cluster of points within $r/10$ of the centre of the circle, then we cannot select 4 points from S (and corresponding points on the circle) having the desired property.

†By the way, H is called the convex hull of S. If the points of S lie on a line, then H will be the shortest line segment containing the points of S. Otherwise, H is a polygon whose vertices are all elements of S and such that all other points in S lie inside or on this polygon.

Solution 2

a) If all the points of S lie on a line L, then choose any 3 of them to be a, b, c. Let A be a point on the circle which meets the perpendicular to L at a. Clearly A is closer to a than to any other point on L, and hence closer than other other point in S. We find B and C in an analogous way.

Otherwise, choose a, b, c from S so that the triangle formed by these points has maximal area. Construct the altitude from the side bc to the point a and extend this line until it meets the circle at A. We claim that A is closer to a than to any other point in S.

Suppose not. Let x be a point in S for which the distance from A to x is less than the distance from A to a. Then the perpendicular distance from x to the line bc must be greater than the perpendicular distance from a to the line bc. But then the triangle formed by the points x, b, c has greater area than the triangle formed by a, b, c, contradicting the original choice of these 3 points. Therefore A is closer to a than to any other point in S.

The points B and C are found by constructing similar altitudes through b and c, respectively.

b) See Solution 1.

4. Let ABC be a triangle with circumradius R, perimeter P and area K. Determine the maximum value of KP/R^3.

Solution 1

Since similar triangles give the same value of KP/R^3, we can fix $R = 1$ and maximize KP over all triangles inscribed in the unit circle. Fix points A and B on the unit circle. The locus of points C with a given perimeter P is an ellipse that meets the circle in at most four points. The area K is maximized (for a fixed P) when C is chosen on the perpendicular bisector of AB, so we get a maximum value for KP if C is where the perpendicular bisector of AB meets the circle. Thus the maximum value of KP for a given AB occurs when ABC is an isosceles triangle. Repeating this argument with BC fixed, we have that the maximum occurs when ABC is an equilateral triangle.

Consider an equilateral triangle with side length a. It has $P = 3a$. It has height equal to $a\sqrt{3}/2$ giving $K = a^2\sqrt{3}/4$. From the extended law of sines, $2R = a/\sin(60)$ giving $R = a/\sqrt{3}$. Therefore the maximum value we seek is

$$KP/R^3 = \left(\frac{a^2\sqrt{3}}{4}\right)(3a)\left(\frac{\sqrt{3}}{a}\right)^3 = \frac{27}{4}.$$

Solution 2

From the extended law of sines, the lengths of the sides of the triangle are $2R\sin A$, $2R\sin B$ and $2R\sin C$. So

$$P = 2R(\sin A + \sin B + \sin C) \text{ and } K = \frac{1}{2}(2R\sin A)(2R\sin B)(\sin C),$$

giving

$$\frac{KP}{R^3} = 4\sin A\sin B\sin C(\sin A + \sin B + \sin C).$$

We wish to find the maximum value of this expression over all $A + B + C = 180°$. Using well-known identities for sums and products of sine functions, we can write

$$\frac{KP}{R^3} = 4\sin A\left(\frac{\cos(B-C)}{2} - \frac{\cos(B+C)}{2}\right)\left(\sin A + 2\sin\left(\frac{B+C}{2}\right)\cos\left(\frac{B-C}{2}\right)\right).$$

If we first consider A to be fixed, then $B + C$ is fixed also and this expression takes its maximum value when $\cos(B-C)$ and $\cos\left(\frac{B-C}{2}\right)$ equal 1; *i.e.* when $B = C$. In a similar way, one can show that for any fixed value of B, KP/R^3 is maximized when $A = C$. Therefore the maximum value of KP/R^3 occurs when $A = B = C = 60°$, and it is now an easy task to substitute this into the above expression to obtain the maximum value of $27/4$.

Solution 3

As in Solution 2, we obtain

$$\frac{KP}{R^3} = 4\sin A \sin B \sin C(\sin A + \sin B + \sin C).$$

From the AM-GM inequality, we have

$$\sin A \sin B \sin C \leq \left(\frac{\sin A + \sin B + \sin C}{3}\right)^3,$$

giving

$$\frac{KP}{R^3} \leq \frac{4}{27}(\sin A + \sin B + \sin C)^4,$$

with equality when $\sin A = \sin B = \sin C$. Since the sine function is concave on the interval from 0 to π, Jensen's inequality gives

$$\frac{\sin A + \sin B + \sin C}{3} \leq \sin\left(\frac{A+B+C}{3}\right) = \sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}.$$

Since equality occurs here when $\sin A = \sin B = \sin C$ also, we can conclude that the maximum value of KP/R^3 is $\frac{4}{27}\left(\frac{3\sqrt{3}}{2}\right)^4 = 27/4$.

5. Let's say that an ordered triple of positive integers (a, b, c) is *n-powerful* if $a \leq b \leq c$, $\gcd(a, b, c) = 1$, and $a^n + b^n + c^n$ is divisible by $a + b + c$. For example, $(1, 2, 2)$ is 5-powerful.

   a) Determine all ordered triples (if any) which are n-powerful for all $n \geq 1$.

   b) Determine all ordered triples (if any) which are 2004-powerful and 2005-powerful, but not 2007-powerful.

   [Note that $\gcd(a, b, c)$ is the greatest common divisor of a, b and c.]

Solution 1

Let $T_n = a^n + b^n + c^n$ and consider the polynomial

$$P(x) = (x - a)(x - b)(x - c) = x^3 - (a + b + c)x^2 + (ab + ac + bc)x - abc.$$

Since $P(a) = 0$, we get $a^3 = (a+b+c)a^2 - (ab+ac+bc)a + abc$ and multiplying both sides by a^{n-3} we obtain $a^n = (a+b+c)a^{n-1} - (ab+ac+bc)a^{n-2} + (abc)a^{n-3}$. Applying the same reasoning, we can obtain similar expressions for b^n and c^n and adding the three identities we get that T_n satisfies the following 3-term recurrence:

$$T_n = (a + b + c)T_{n-1} - (ab + ac + bc)T_{n-2} + (abc)T_{n-3}, \text{ for all } n \geq 3.$$

From this we see that if T_{n-2} and T_{n-3} are divisible by $a + b + c$, then so is T_n. This immediately resolves part (b)—there are no ordered triples which are 2004-powerful and 2005-powerful, but not 2007-powerful—and reduces the number of cases to be considered in part (a): since all triples are 1-powerful, the recurrence implies that any ordered triple which is both 2-powerful and 3-powerful is n-powerful for all $n \geq 1$.

Putting $n = 3$ in the recurrence, we have

$$a^3 + b^3 + c^3 = (a + b + c)(a^2 + b^2 + c^2) - (ab + ac + bc)(a + b + c) + 3abc$$

which implies that (a, b, c) is 3-powerful if and only if $3abc$ is divisible by $a + b + c$. Since

$$a^2 + b^2 + c^2 = (a + b + c)^2 - 2(ab + ac + bc),$$

(a, b, c) is 2-powerful if and only if $2(ab + ac + bc)$ is divisible by $a + b + c$.

Suppose a prime $p \geq 5$ divides $a + b + c$. Then p divides abc. Since $\gcd(a, b, c) = 1$, p divides exactly one of a, b or c; but then p doesn't divide $2(ab + ac + bc)$.

Suppose 3^2 divides $a + b + c$. Then 3 divides abc, implying 3 divides exactly one of a, b or c. But then 3 doesn't divide $2(ab + ac + bc)$.

Suppose 2^2 divides $a + b + c$. Then 4 divides abc. Since $\gcd(a, b, c) = 1$, at most one of a, b or c is even, implying one of a, b, c is divisible by 4 and the others are odd. But then $ab + ac + bc$ is odd and 4 doesn't divide $2(ab + ac + bc)$.

So if (a, b, c) is 2- and 3-powerful, then $a + b + c$ is not divisible by 4 or 9 or any prime greater than 3. Since $a + b + c$ is at least 3, $a + b + c$ is either 3 or 6. It is now a simple matter to check the possibilities and conclude that the only triples which are n-powerful for all $n \geq 1$ are $(1, 1, 1)$ and $(1, 1, 4)$.

Solution 2

Let p be a prime. By Fermat's Little Theorem,

$$a^{p-1} \equiv \begin{cases} 1 \pmod{p}, & \text{if } p \text{ doesn't divide } a; \\ 0 \pmod{p}, & \text{if } p \text{ divides } a. \end{cases}$$

Since $\gcd(a, b, c) = 1$, we have that $a^{p-1} + b^{p-1} + c^{p-1} \equiv 1, 2$ or $3 \pmod{p}$. Therefore if p is a prime divisor of $a^{p-1} + b^{p-1} + c^{p-1}$, then p equals 2 or 3. So if (a, b, c) is n-powerful for all $n \geq 1$, then the only primes which can divide $a + b + c$ are 2 or 3.

We can proceed in a similar fashion to show that $a + b + c$ is not divisible by 4 or 9.

Since

$$a^2 \equiv \begin{cases} 0 \pmod{4}, & \text{if } p \text{ is even;} \\ 1 \pmod{4}, & \text{if } p \text{ is odd} \end{cases}$$

and a, b, c aren't all even, we have that $a^2 + b^2 + c^2 \equiv 1, 2$ or $3 \pmod{4}$.

By expanding $(3k)^3$, $(3k + 1)^3$ and $(3k + 2)^3$, we find that a^3 is congruent to 0, 1 or -1 modulo 9. Hence

$$a^6 \equiv \begin{cases} 0 \pmod{9}, & \text{if 3 divides } a; \\ 1 \pmod{9}, & \text{if 3 doesn't divide } a. \end{cases}$$

Since a, b, c aren't all divisible by 3, we have that $a^6 + b^6 + c^6 \equiv 1, 2$ or $3 \pmod{9}$.

So $a^2 + b^2 + c^2$ is not divisible by 4 and $a^6 + b^6 + c^6$ is not divisible by 9. Thus if (a, b, c) is n-powerful for all $n \geq 1$, then $a + b + c$ is not divisible by 4 or 9. Therefore $a + b + c$ is either 3 or 6 and checking all possibilities, we conclude that the only triples which are n-powerful for all $n \geq 1$ are $(1, 1, 1)$ and $(1, 1, 4)$.

See Solution 1 for the (b) part.

GRADER'S REPORT

Each question was worth a maximum of 7 marks. Every solution on every paper was graded by two different markers. If the two marks differed by more than one point, the solution was reconsidered until the difference was resolved. If the two marks differed by one point, the average was used in computing the total score. The top papers were then reconsidered until the committee was confident that the prize-winning contestants were ranked correctly.

The various marks assigned to each solution are displayed below, as a percentage. As described above, fractional scores are possible, but for the purpose of this table, marks are rounded up. So, for example, 54.7% of the students obtained a score of 6.5 or 7 on the first problem. This indicates that on 54.7% of the papers, at least one marker must have awarded a 7 on question #1.

Marks	**#1**	**#2**	**#3**	**#4**	**#5**
0	9.3	4.0	53.3	20.0	48.0
1	6.7	2.7	12.0	16.0	41.3
2	2.7	10.7	12.0	1.3	2.7
3	1.3	22.7	4.0	1.3	2.7
4	4.0	18.7	10.7	6.7	0.0
5	5.3	10.7	2.7	17.3	1.3
6	16.0	10.7	5.3	8.0	1.3
7	54.7	20.0	0.0	29.3	2.7

At the outset our marking philosophy was as follows: A score of 7 was given for a completely correct solution. A score of 6 indicated a solution which was essentially correct, but with a very minor error or omission. Very significant progress had to be made to obtain a score of 3. Even scores of 1 or 2 were not awarded unless some significant work was done. Scores of 4 and 5 were reserved for special situations. This approach had to be modified somewhat for the questions with more than one part.

PROBLEM 1

This problem was very well done. Although there were a few slightly different ways to proceed (some students used induction, for example), every solution essentially involves enumerating the number of possible ways that the path can get from one row to the next.

PROBLEM 2

This problem was fairly well done with most students making significant progress on at least one of the two parts, usually (a). Three marks were awarded for a correct solution to the (a) part. There were many different ways to proceed here and the four official solutions provide a representative sample. The most common approach was to use AM-GM in a manner similar to Solution 3. One mark was deducted if students didn't show that the inequality was strict. Four marks were awarded for a correct solution to (b). Again there are many different ways to proceed, but the students had a more difficult time writing clear solutions to this part.

PROBLEM 3

This problem proved to be quite difficult and no student obtained a perfect score. Four marks were awarded for solving (a) and three marks were awarded for (b). About 12 students managed to solve the (b) part, but only 4 were able to provide a complete proof for (a). Many students made partial progress on the (a) part only to find that their argument didn't cover all possible situations. This was the most challenging question to grade.

PROBLEM 4

This geometry problem was quite well done. About half of the contestants realized that the maximum value occurred when the triangle was equilateral, but it was necessary to prove this to obtain full marks. Two students gave geometric arguments similar to Solution 1. Most students expressed KP/R^3 in terms of trig functions (as in Solutions 2, but there were many variations) and attempted to maximize the expression over all possible angles. There are many ways to do this, but some care had to be taken. Solutions 2 and 3 show two of the better approaches.

PROBLEM 5

Few students made significant progress on this challenging problem. Five marks were awarded for the (a) part and two marks for (b). The four students who attained high marks on this question all used an approach similar to Solution 1. One mark was given to students who found a solution by inspection.