SULLIVAN MACDONALD, University of Toronto

Progress toward the Krzyz conjecture

The Krzyz conjecture is a long-standing open problem in complex analysis. Despite initially appearing simpler than related problems which have since been solved, such as the Bieberbach conjecture (now de Branges' Theorem), it remains open. If D is the unit disc and $\mathcal{B}^* = \{f \in \mathrm{Hol}(D) \mid 0 < |f| \leq 1 \text{ in } D\}$, it states that (1) $\sup_{f\in\mathcal{B}^*} |f^{(n)}(0)|/n! = 2/e$ for any $n \in \mathbb{N}$, and (2) the supremum is attained only by functions of the form $e^{i\theta} f_0(e^{i\eta} z)$, where $\theta, \eta \in \mathbb{R}$ and $f_0(z) = \exp((z^n - 1)/(z^n + 1))$.

In this talk we present recent work on the conjecture. Using techniques from classical harmonic analysis, we find new constraints on the singular inner functions which attain the supremum. It has long been known that extremal functions in $\mathcal{B}^*$ for the nth coefficient exist and are of the form

$$f(z) = \exp\left(\sum_{j=1}^{N} \lambda_j \frac{e^{i\theta_j} z - 1}{e^{i\theta_j} z + 1}\right)$$

for $N \leq n$, positive $\lambda_1, \ldots, \lambda_N$, and distinct $\theta_1, \ldots, \theta_N \in [0, 2\pi)$. Using oscillatory integral methods, we show that $N \geq c\, n$ for a universal constant $c > 0$. This marks modest progress toward proving the expected $N = n$. Various other new properties of extremal functions and their consequences will also be discussed.

Furthermore, we will report on progress related to other aspects of the conjecture.