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Explicit Estimates for the Size of the Markoff mod p Cage

The Markoff equation is given by $X^2 + Y^2 + Z^2 = 3XYZ$ and the positive integer solutions to this equation are called Markoff triples. Involutions can be defined on the set of Markoff triples by fixing two coordinates and finding the other root of the resulting quadratic. The group generated by these involutions and coordinate permutations, called the Vieta group, acts on the set of Markoff triples, and the graph defined by this action is known to be connected. In 1991, Baragar conjectured that the same holds modulo any prime p . Since the action of the Vieta group commutes with reduction mod p , this would imply that every mod p solution to the Markoff equation has an integer lift. In 2015, Bourgain, Gamburd, and Sarnak made progress toward this conjecture by proving that a subgraph called the cage is always connected. Computations suggest that roughly 80% of triples lie in the cage, but explicit bounds have been difficult to find. We provide an explicit lower bound for the size of the cage for primes p where $p + 1$ has sufficiently large valuation. We also estimate the size of the cage for all primes in terms of the Euler totient function under a reasonable heuristic assumption. Our work also yields practical conditions for a Markoff triple to lie in the cage.