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The continuous primitive integral

The space of all Schwartz distributions, $\mathcal{D}'(\mathbb{R})$, is too large for a viable theory of integration. However, by looking for appropriate Banach spaces of distributions we can define integration processes that have many useful properties.

Primitives for the Lebesgue integral are absolutely continuous. If we take primitives to be merely continuous we obtain an integral that includes the Lebesgue and Henstock–Kurzweil integrals. Define the set of primitives $\mathcal{B}_c(\mathbb{R}) = \{F : \overline{\mathbb{R}} \to \mathbb{R} \mid F(-\infty) = 0, F$ is continuous on $\overline{\mathbb{R}}\}$, where $\overline{\mathbb{R}} = [-\infty, \infty]$ is the extended real line. Then $\mathcal{B}_c(\mathbb{R})$ is a Banach space under the uniform norm. Define the space of integrable distributions by taking the distributional derivative of the primitives: $\mathcal{A}_c(\mathbb{R}) = \{f \in \mathcal{D}'(\mathbb{R}) \mid f = F'$ for some $F \in \mathcal{B}_c(\mathbb{R})\}$. The definition of the integral is based on the fundamental theorem of calculus: If $f \in \mathcal{A}_c(\mathbb{R})$ then $\int_a^b f = F(b) - F(a)$ where $f = F'$ for a unique primitive $F \in \mathcal{B}_c(\mathbb{R})$. The Alexiewicz norm of f is $\|f\| = \|F\|_\infty$ and this makes $\mathcal{A}_c(\mathbb{R})$ into a Banach space that is isometrically isomorphic to $\mathcal{B}_c(\mathbb{R})$. This space is the completion of $L^1(\mathbb{R})$ in the Alexiewicz norm and is the smallest Banach space that contains all functions that have conditionally convergent integrals. Features useful in applications, such as a Hölder inequality, convergence theorems, convolution, and integration by parts will be discussed.