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Discretization of adapted functions

We say that a family $\{\psi_\gamma\}_{\gamma\in\Gamma} \subset L^2(\mathbf{R}^d)$ is almost-orthogonal if there is a finite R so that, for all finite $\mathcal{F} \subset \Gamma$ and all linear combinations $\sum_{\gamma\in\mathcal{F}} \lambda_\gamma \psi_\gamma$, where $\{\lambda_\gamma\}_{\gamma\in\mathcal{F}} \subset \mathbf{C}$,

$$\left\| \sum_{\mathcal{F}} \lambda_\gamma \psi_\gamma \right\|_2 \leq R \left(\sum_{\mathcal{F}} |\lambda_\gamma|^2 \right)^{1/2}.$$

The least R for which this holds is called the family's almost-orthogonality constant, denoted $\|\{\psi_\gamma\}_{\gamma\in\Gamma}\|_{AO(\Gamma)}$. The almost-orthogonality constant can be used to quantify how far a family is from having certain useful properties (orthonormality, being a frame, etc.). We show:

Theorem. Let $0 < \alpha, \tau \leq 1$. Let $\{f^{(Q)}\}_{Q\in\mathcal{D}}$ be a family of functions indexed over the dyadic cubes $\mathcal{D}$ in $\mathbf{R}^d$, satisfying: a) $\forall Q \in \mathcal{D}$, supp $f^{(Q)} \subset \overline{Q}$; b) $\forall x, x' \in \mathbf{R}^d (|f^{(Q)}(x) - f^{(Q)}(x')| \leq (|x - x'|/\ell(Q))^\alpha$, where $\ell(Q)$ is Q's sidelength. For each $Q \in \mathcal{D}$ let $\mathcal{G}(Q)$ be a set of disjoint dyadic cubes J such that $\ell(J) \leq \tau\ell(Q)$ and $\cup_{\mathcal{G}(Q)} J = Q$. Set

$$f^{(Q)}_{\mathcal{G}(Q)} := \sum_{J\in\mathcal{G}(Q)} f^{(Q)}_J \chi_J,$$

where $f^{(Q)}_J$ means $f^{(Q)}$'s average over J. There is a constant $C(\alpha, d)$, depending only on α and d, so that

$$\left\| \left\{ \frac{f^{(Q)} - f^{(Q)}_{\mathcal{G}(Q)}}{|Q|^{1/2}} \right\}_{Q\in\mathcal{D}} \right\|_{AO(\mathcal{D})} \leq C(\alpha, d)\tau^\alpha,$$

where $|Q| =$ the measure of Q.

What this means is that, if we apply sufficiently fine dyadic stopping times to the functions in $\{f^{(Q)}/|Q|^{1/2}\}_{Q\in\mathcal{D}}$, the resulting family $\{f^{(Q)}_{\mathcal{G}(Q)}/|Q|^{1/2}\}_{Q\in\mathcal{D}}$ is close to $\{f^{(Q)}/|Q|^{1/2}\}_{Q\in\mathcal{D}}$ in the almost-orthogonal sense. If time permits we will say a few words about a companion result (in different ways stronger and weaker) for such families when $\alpha = 1$.