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When do interacting organisms gravitate to the vertices of a regular simplex?

Flocking and swarming models often assume that organisms interact through a force which is attractive over large distances yet repulsive at short distances. Assume this force is given as a difference of power laws and normalized so that its unique minimum occurs at unit separation. For a range of exponents corresponding to mild repulsion and strong attraction, we show that the minimum energy configuration is uniquely attained — apart from translations and rotations — by equidistributing the organisms over the vertices of a regular top-dimensional simplex (i.e. an equilateral triangle in two dimensions and regular tetrahedron in three).

If the attraction is not assumed to be strong, we show these configurations are at least local energy minimizers in the relevant d_∞ metric from optimal transportation, as are all of the other uncountably many unbalanced configurations with the same support. We infer the existence of phase transitions.

One ingredient from the proof will be described in another session, namely the establishment of a simple isodiametric variance bound which characterizes regular simplices: it shows that among probability measures on $\mathbf{R}^n$ whose supports have at most unit diameter, the variance around the mean is maximized precisely by those measures which assign mass $1/(n+1)$ to each vertex of a (unit-diameter) regular simplex.

Based on preprint with Tongseok Lim at `https://arxiv.org/abs/1907.13593`