Fractal Geometry, Analysis, and Applications
Géométrie fractale, analyse et applications

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WILLIAM C. ABRAM, Hillsdale College

Intersections of Cantor Sets and Self-Similarity

We study finite intersections of multiplicative translates of p-adic Cantor sets. For $p = 3$, this is motivated by a problem of Erdős on the base 3 expansions of powers of 2. Consider the discrete dynamical system on the 3-adic integers $\mathbb{Z}_3$ given by multiplication by 2. The exceptional set $\mathcal{E}(\mathbb{Z}_3)$ is defined to be the set of all elements of $\mathbb{Z}_3$ whose forward orbits under this action intersect the 3-adic Cantor set $\Sigma_{3,\bar{2}}$ (of 3-adic integers that omit the digit 2) infinitely many times. This set is conjectured to have Hausdorff dimension 0, and attempts to prove this conjecture have led to the study of many interesting families of intersections of Cantor sets. These intersection sets are fractals whose points have 3-adic expansions describable by labeled paths in a finite automaton, whose Hausdorff dimension is exactly computable and is of the form $log_3(\beta)$ where β is a real algebraic integer.

The theoretical framework that we have developed to study intersections of Cantor sets, including the idea of a path set and of p-adic path set fractals, has found application to the study of multi-layer cellular networks. I will also discuss current work using this framework to study the self-similarity of intersections and unions of translations of Cantor sets.

BALÁZS BÁRÁNY, Budapest University of Technology, MTA-BME Stochastics Research Group

On the Hausdorff dimension of self-affine sets and measures

In the last few years considerable attention has been paid for the dimension theory of self-affine sets and measures, furthermore new methods and new techniques appeared in this field. The Furstenberg measure, which is the stationary measure induced by the cocycle of the matrices, plays an important role. We will show that self-affine measures satisfy the Ledrappier-Young formula, and from the dimension of the induced Furstenberg measure we can conclude to the dimension of the self-affine measure. This two facts allows us to give two different, almost every type results on the dimension w.r.t. the matrices.

The talk is based on my joint works with Antti Käenmäki, Henna Koivusalo, Michał Rams and Károly Simon.

ILIA BINDER, University of Toronto

Multifractal spectrum of SLE boundary collisions.

I will discuss the multifractal spectrum of the intersection of chordal SLE$_\kappa$ curves with the real line. For $\kappa > 4$, this intersection is a random fractal of almost sure Hausdorff dimension $\min\{2 - 8/\kappa, 1\}$. We study the random sets of points at which the curve collides with the real line at a specified "angle" (or, equivalently, the local dimension of harmonic measure is prescribed) and compute an almost sure dimension spectrum describing the metric size of these sets. The talk is based on a joint work with Tom Alberts (Utah) and Fredrik Viklund (KTH).

TRUBEE DAVISON, Unaffiliated

A Positive Operator-Valued Measure Associated to an Iterated Function System

Given an iterated function system (IFS) on a complete and separable metric space Y, there exists a unique compact subset $X \subseteq Y$ satisfying a fixed point relation with respect to this IFS. This subset is called the attractor set, or fractal set, associated to the IFS. The attractor set supports a specific Borel probability measure, called the Hutchinson measure, which itself satisfies a fixed point relation. P. Jorgensen generalized the Hutchinson measure to a projection-valued measure, under the assumption that the IFS does not have essential overlap. The situation when the IFS exhibits essential overlap has also been studied by Jorgensen and colleagues. We build off their work to generalize the Hutchinson measure to a positive operator-valued measure

for a general IFS, that may exhibit essential overlap. We also discuss Naimark's dilation theorem with respect to this positive operator-valued measure.

IGNACIO GARCÍA, University of Waterloo
Assouad dimensions of complementary sets

Given a positive, decreasing sequence a, whose sum is L, we consider all the closed subsets of $[0, L]$ such that the lengths of their complementary open intervals are in one to one correspondence with the sequence a. The sets in this class have zero Lebesgue measure. In this talk I'm going to discuss the possible values that Assouad-type dimensions can attain for this class of sets. In many cases, the set of attainable values is a closed interval whose endpoints we determine. This is joint work with Kathryn Hare and Franklin Mendivil.

KEVIN HARE, University of Waterloo
Families of self-affine maps

Let $f_1, f_2, \ldots, f_n$ be a set of contraction maps. We define the IFS based on $f_1, f_2, \ldots, f_n$ as the unique non-trivial compact operator K such that $K = \cup f_i(K)$. In this talk we consider the very simple IFS coming from the two contraction maps, $f_1(\vec{v}) = A\vec{v} - \vec{a}$ and $f_2(\vec{v}) = A\vec{v} + \vec{a}$. Here we will consider the structure of the IFS based upon the matrix A, considering such things as for which A is the IFS connected, or totally disconnected, or having interior? There is a surprising rich structure to these questions.

HERB KUNZE, University of Guelph
Star-Shaped Set Inversion Map Fractals

Inversion of points in the plane with respect to a circle has been of interest to geometers, including those working in fractals. Indeed, in *The Fractal Geometry of Nature*, Mandelbrot discusses successive inversion with respect to a family of circles. Some relatively recent work illustrates the fractal attractors generated by systems of circle and/or star-shaped set inversion maps. It appears that the handful of papers in the literature focus on the graphical beauty of the sets, without any careful discussion of the underlying mathematical machinery. In this talk, we establish that such systems can be cast in terms of contractive maps on an appropriate complete metric space. We show some examples and, for fun, demonstrate that the LIFSM framework for signals can readily be adapted to use such maps. (This talk is based on the MSc work of B. Boreland.)

LUKE ROGERS, University of Connecticut
Spectral properties of pseudodifferential operators on the Sierpinski Gasket

The celebrated Szegö limit theorem states that

$$\lim_{n\to\infty} \frac{1}{n+1} \log\det P_n[f]P_n = \int_0^{2\pi} \log f(\theta)\, \frac{d\theta}{2\pi}$$

where $[f]$ is the operator of multiplication by a positive $C^{1+\alpha}$ function f and P_n is the projection onto the span of $\{e^{ik\theta}, 0 \leq k \leq n\}$ in L^2 of the unit circle. Later generalizations allow more complicated operators than $[f]$.

In joint work with M. Ionescu and K. Okoudjou we consider an analogue of this result on the Sierpinski Gasket in which $[f]$ is replaced by a pseudo-differential operator, the projection is onto the eigenspaces of L^2 with eigenvalues less than Λ and the limit is taken as $\Lambda \to \infty$. For a suitable class of pseudo-differential operators the result gives asymptotics of the operator from its symbol.

DANIEL SLONIM, Purdue
Path Sets and Interleaving

I will first give a brief overview of how questions in number theory and fractal geometry led to a discussion of sequences generated by infinite walks on graphs. Given the right structure, a graph presents a set of sequences, called a "path set." A path set $\mathcal{P}$ can be associated to a fractal set of p-adic integers, and the toplogical entropy of $\mathcal{P}$ can be used to calculate the Hausdorff dimension of its associated fractal set. We will look at interleaving and related operations on these path sets, discuss the effect these operations have on the graphs presenting the path sets, and discuss an algorithm for detecting path sets that are irreducible with respect to the interleaving operations.

JÓZSEF VASS, York University
Fractal Potentials of the Laplace and Wave Equations

The invariant measure introduced by Hutchinson is the fixed point of a contractive transfer operator, and it is supported on an IFS fractal. The Laplacian and the d'Alembertian operate on functions in Poisson's equation, in space and spacetime respectively. With appropriate boundary conditions at infinity, this equation induces a bijection between potentials and probability measures, which becomes an isometric isomorphism under certain metrics. It enables us to show the existence of an "invariant potential" of the appropriately generalized transfer operator, which corresponds uniquely to the invariant measure. This invariant potential is also referred to as a "fractal potential", a weakly harmonic function that satisfies the Laplace or Wave equations almost everywhere. Its singularities correspond to the support of the invariant measure, required to have vanishing Lebesgue measure.

ANDREW VINCE, University of Florida
Fractal Transformations

Fractal transformations are natural maps associated with an iterated function system (IFS). A *fractal transformation* is basically a transformation that takes each point in the attractor of one IFS to a point with the same address in another IFS. A precise definition, examples, results, and applications will be discussed.

ALDEN WALKER, Center for Communications Research
Circle actions on the boundary of Schottky space

To a complex parameter c, we associate the two-generator iterated function system f(x)=cz-1, g(z)=cz+1. I'll describe how the IFS for certain parameters (those on the boundary of the connectedness locus) can give rise to circle actions. A finite amount of data encoded in these circle actions describes the set of cut points in the limit set of the IFS. In addition, these circle actions can be thought of as double covers of Lorenz maps and generalizations. This talk should be broadly accessible, and pictures will be provided. This is joint work with Danny Calegari, building on previous work with Danny Calegari and Sarah Koch.

BOMING YU, Huazhong University of Science and Technology
A review on the fractal geometry theory for porous media and its applications

Available data have been shown that the microstructures of naturally formed porous media such as soil, rocks, sandstones, oil/gas//water reservoirs, biological tissue and organics, etc. are fractal objects and can be described by the fractal geometry theory and technique. This presentation attempts to review and summarize the progresses on research in the area of the fractal geometry theory and technique for porous media. Then, review and summary are presented for the progresses on research of the applications of the theory and technique in the areas such as transport properties of fractal porous media regarding the thermal conductivities, permeabilities, gas diffusivity and imbibitions based on the fractal geometry theory and technique for porous media. Finally, a few of comments are made with respect to the theoretical developments and applications in the future.

Keywords: Fractal, Transport properties, Porous media.

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