Non-Linear PDE of Mathematical Physics
Les EPD nonlinéaires venant de la physique mathématique
(Org: **Stan Alama** and/et **Lia Bronsard** (McMaster))

GINO BIONDINI, State University of New York at Buffalo
The integrable nature of modulational instability

We investigate the nonlinear stage of the modulational instability (MI) by characterizing the IVP for the focusing NLS equation with non-zero boundary conditions (NZBC) at infinity, using the recently formulated inverse scattering transform (IST) for this problem. While the linearization of NLS ceases to be valid when perturbations have grown sufficiently large, the IST holds at all times, and therefore provides the best way to study the nonlinear stage of MI. First we study the scattering problem with piecewise constant ICs which are generalizations to NZBC of a potential well and barrier, and we obtain several results. We prove that there are arbitrarily small perturbations of the constant background for which there are discrete eigenvalues, which shows that no area theorem is possible for the NLS equation with NZBC. We prove that there there is a large class of perturbations for which no discrete eigenvalues are present, which shows that solitons cannot be the primary vehicle for the manifestation of MI, contrary to a recent conjecture. Finally, we compute the small-deviation limit of IST and we compare it with the linearization of NLS. This allows us to identify the precise nonlinear analogue of the unstable Fourier modes within the IST. These are the Jost eigenfunctions for values of the scattering parameter in a finite interval of the imaginary axis around the origin. Importantly, this shows that the IST contains an automatic mechanism for the saturation of the MI.

STEFANELLA BOATTO, Universidade Federal do Rio de Janeiro

ALMUT BURCHARD, University of Toronto
Geometric stability of the Coulomb energy

The Coulomb energy of a positive charge distribution increases under symmetrization. In particular, if the charge is uniformly distributed on some set, then the energy is maximized (among sets of given volume) by balls. I will present recent work with Greg Chambers where we show that near-maximizers are close to balls.

YURI CHER, University Of Toronto
On a class of Generalized Derivative Nonlinear Schrödinger Equations

We study a class of generalized Derivative Nonlinear Schrödinger (gDNLS) equations of the form $i\psi_t + \psi_{xx} + i|\psi|^{2\sigma}\psi_x = 0$ with $\sigma > 1$. When $\sigma = 1$, this equation reduces to the canonical DNLS equation that arises from magnetohydrodynamics as well as in studies of ultrashort optical pulses. The DNLS shares the same scaling properties (L^2 critical) as the quintic Nonlinear Schrödinger equation which is known to have finite time blow-up solutions. However, the long time existence/possible occurrence of singularities for the DNLS equation remains an open problem. Recent numerical studies of the gDNLS for a range of values of $\sigma \in (1, 2]$ indicate that finite time blow-up may occur and give precisely the local structure of the blow-up solutions in terms of the blow-up rate and a universal profile Q depending only on the strength of the nonlinearity σ. This complex valued profile is a solution of a nonlinear elliptic ODE with 2 real valued parameters a and b with an integral constraint.

Using methods of asymptotic analysis, we study the deformation of this profile and the parameters as the nonlinearity σ tends to 1. We find that Q tends to the lump soliton of DNLS while the parameter a tends to 0 like a power law in $(\sigma - 1)$. We compare our results to a numerical integration of the nonlinear elliptic ODE using continuation methods. This is a joint work with G. Simpson and C. Sulem.

ALEXEI CHEVIAKOV, University of Saskatchewan
Nonlinear Equations for Finite-Amplitude Wave Propagation in Fiber-Reinforced Hyperelastic Media

Various composite materials, including biological tissues, are modeled as nonlinear elastic materials reinforced with elastic fibers. We consider the full set of dynamic equations for finite deformations of incompressible hyperelastic solids reinforced by a single fiber family. Using finite-amplitude wave propagation ansätze compatible with the incompressibility condition, we derive the corresponding nonlinear and linear wave equations. Properties of these equations, including point symmetries, are studied. Exact time-dependent solutions are derived in several physical situations, including the evolution of pre-strained configurations and traveling waves.

MARINA CHUGUNOVA, Claremont Graduate University
Nonnegative weak solutions for a degenerate system modeling the spreading of surfactant on thin films

Depending on the parameter range, we prove local and global in time existence of non- negative weak solutions to a coupled system of two degenerate parabolic equations. This system models the spreading of an insoluble surfactant on a thin liquid film. The model includes gravity, surface tension, capillarity effects, and van der Waals forces. The surface diffusion coefficient is not assumed constant and depends on the surfactant concentration.

Joint work with: Roman M. Taranets, UCLA

ANDRES CONTRERAS, Fields

WALTER CRAIG, McMaster University and The Fields Institute
Birkhoff normal form for wave equation null forms

Theorems on global existence of solutions of nonlinear wave equations in $\mathbb{R}^n$ depend upon a competition between the time decay of solutions and the degree of the nonlinearity. Decay estimates are more effective when inessential nonlinear terms are able to be removed through a well-chosen transformation. In this talk, we construct Birkhoff normal forms transformations for the class of wave equations which are Hamiltonian PDEs and null forms, giving a new proof via canonical transformations of the global (and 'almost global') existence theorems for null form wave equations of S. Klainerman and J. Shatah. I will also describe the potential further applications to PDEs of this technique of Hamiltonian dynamics. These results are work-in-progress with A. French and C.-R. Yang.

BERNARDO GALVAO-SOUSA, University of Toronto
Accelerating Fronts in Semilinear Wave Equations

I will study dynamics of interfaces in solutions of the equation $\varepsilon\,\Box u + \frac{1}{\varepsilon} f_\varepsilon(u) = 0$, for f_ε of the form $f_\varepsilon(u) = (u^2-1)(2u-\varepsilon\kappa)$, for $\kappa \in \mathbb{R}$, as well as more general, but qualitatively similar, nonlinearities. I will show that for suitable initial data, solutions exhibit interfaces that sweep out timelike hypersurfaces of mean curvature proportional to κ.

This is a joint work with Robert Jerrard (University of Toronto).

QI GAO, NCTS, National Taiwan University
Symmetric vortices of two-component Ginzburg-Landau system

We consider symmetric vortex solutions in the plane $\mathbb{R}^2$, $\Psi = (\psi_+(x), \psi_-(x)) = (f_+(r)e^{in_+\theta}, f_-(r)e^{in_-\theta})$, with given degrees $n_\pm \in \mathbb{Z}$, and prove existence, uniqueness, and asymptotic behavior of solutions as $r \to \infty$. We also consider the monotonicity properties of solutions, and exhibit parameter ranges in which both vortex profiles f_+, f_- are monotone, as well as parameter regimes where one component is non-monotone. The qualitative results are obtained by means of a sub- and supersolution construction and a comparison theorem for elliptic systems. This is joint work with S. Alama.

TIZIANA GIORGI, New Mexico State University
Field-induced smectic phases in liquid crystals

We will discuss a phenomenological Landau energy introduced by Vaupotič and Čopič to study the onset of layer-undulated structures, induced by an applied electric field, in bent-core molecules liquid crystals. We will also present an analysis of the chevron structure in Smectic A liquid crystals due to magnetic fields, as modelled by a Chen-Lubensky energy. We will refer to joint work with Carlos García-Cervera and Sookyung Joo.

DMITRY GOLOVATY, The University of Akron
Solutions of Ginzburg-Landau equations in the presence of weak electric currents

I will discuss a Ginzburg-Landau model that allows for weak electric currents and in which the magnetic field is neglected. I will demonstrate the existence of a steady-state solution in a vicinity of the purely superconducting state and show that this solution is linearly stable.

ROBERT L JERRARD, University of Toronto
Existence and uniqueness of minimizers of general least gradient problems

Motivated by problems arising in conductivity imaging, we prove existence, uniqueness, and comparison theorems - under certain sharp conditions - for minimizers of the general least gradient problem

$$\inf_{u\in BV_f(\Omega)}\int_\Omega \varphi(x,Du),$$

where $f:\partial\Omega\to\mathbb{R}$ is continuous,

$$BV_f(\Omega):=\{v\in BV(\Omega):\quad \forall x\in\partial\Omega,\ \ \lim_{r\to 0}\ \textit{ess sup}_{y\in\Omega,|x-y|<r}|f(x)-v(y)|=0\ \}$$

and $\varphi(x,\xi)$ is a function that, among other properties, is convex and homogeneous of degree 1 with respect to the ξ variable. In particular we prove that if $a\in C^{1,1}(\Omega)$ is bounded away from zero, then minimizers of the weighted least gradient problem $\inf_{u\in BV_f}\int_\Omega a|Du|$ are unique in $BV_f(\Omega)$. We construct counterexamples to show that the regularity assumption $a\in C^{1,1}$ is sharp, in the sense that it can not be replaced by $a\in C^{1,\alpha}(\Omega)$ with any $\alpha<1$. This is joint work with Amir Moradifam and Adrian Nachman.

LYUDMILA KOROBENKO, McMaster University
Infinitely degenerate elliptic equations and non doubling metrics

The talk is about regularity of weak solutions to second order infinitely degenerate elliptic equations. It is known that regularity of weak solutions can be studied by studying properties of certain metric spaces associated to the operator, namely subunit metric spaces. The problem arising in the infinitely degenerate case is that the metric is such that the Lebesgue measure of subunit balls is non doubling. This makes a well developed theory of homogeneous metric spaces not applicable. However, with some additional assumptions continuity of weak solutions can be obtained even in the non doubling case. The question that remains open is to prove certain assumptions (such as Sobolev inequality) for at least some types of non doubling metric spaces associated to infinitely degenerate operators.

XAVIER LAMY, McMaster and Université de Lyon
Uniaxial symmetry and 'biaxial escape' in nematic liquid crystals

In nematics , molecules tend to align in some preferred direction. The local alignment may be biaxial, or present uniaxial symmetry. It has long been predicted that 'biaxial escape' should occur in some particular regimes. We will present some new insight on the uniaxiality constraint, independently of the regime. We also present a rigorous proof of 'biaxial escape' in the low temperature regime (joint work with A.Contreras).

ROBERT MCCANN, University of Toronto
The spectrum for a family of fourth order diffusions near the self-similar attractor

The thin-film and quantum drift-diffusion equations belong to a fourth-order family of evolution equations proposed by Denzler and myself as analogous to the (second-order) porous medium family. They are 2-Wasserstein gradient flows of the generalized Fisher information (just as Otto showed the porous medium to be the 2-Wasserstein gradient flow of the Reyni entropy). In this talk we describe the linearization of the fourth-order dynamics around the self-similar solution. We diagonalize this linearization by relating it to analogous problem for the porous medium equation. This yields information about the leading- and higher-order asymptotics of the fourth-order flows on R^n which — outside of special cases — were inaccessible previously. These results were obtained jointly with Christian Seis.

DMITRY PELINOVSKY, McMaster University
Orbital stability of periodic waves and black solitons in the cubic defocusing NLS equation

Periodic waves of the one-dimensional cubic defocusing NLS equation are considered. Using tools from integrability theory, these waves have been shown to be linearly stable and their Floquet-Bloch spectrum has been explicitly computed. We combine here the first four conserved quantities of the NLS equation to give a direct proof that small amplitude periodic waves are orbitally stable with respect to subharmonic perturbations, with period equal to an integer multiple of the period of the wave. We also show that the black soliton is an unconditional minimizer of a fourth-order energy functional, which gives a simple proof of orbital stability with respect to perturbations in $H^2(\mathbb{R})$.

MARY PUGH, University of Toronto
Special Solutions in Smectic Electroconvection

We discuss electroconvection in a free submicron-thick liquid crystal film in an annular geometry. The film is flat in the xy plane; seen from above it looks like a DVD. (Seen from above, it has two boundaries: concentric circles.) A voltage is applied across the film, from the inner boundary to the outer boundary; this voltage provides a convective forcing. Because of the annular geometry, the dynamics are periodic in the azimuthal direction and the only boundaries are those at which the convective forcing is applied. The liquid crystal is in smectic A phase, forming a nearly-perfect two-dimensional fluid because the film does not change thickness, even while flowing. Also, the inner electrode can be rotated and so the experiment can be used to study the interplay between a stabilizing force applied via the boundary (Couette shear) and convection. We present numerical simulations of special solutions such as convection cells, oscillatory convection cells, undulating convection cells, and localized vortex solutions. This is joint work with Stephen Morris (Physics, University of Toronto).

IHSAN TOPALOGLU, McMaster University
Nonlocal attractive-repulsive interaction energies of binary densities

In this talk I will consider the minimization of nonlocal interaction energies of the form

$$E[\rho] = \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} K(x-y)\rho(x)\rho(y)\,dxdy$$

over the set of admissible functions $\{\rho \in L^1(\mathbb{R}^n, \{0,1\})\colon \|\rho\|_{L^1} = 1\}$ where the interaction potential K is of the form of a power-law with attractive and repulsive components. This type of energies arise naturally in descriptions of systems of interacting particles, as well as continuum descriptions of systems with long-range interactions and are used in modelling collective behavior of many-agent systems, granular media and self-assembly of nanoparticles. The additional nonconvex constraint $\rho \in \{0,1\}$, on one hand, poses new challenges such as the existence of minimizers. On the other hand, with this constraint the first and second variations of the energy E can simply be expressed explicitly as conditions on the boundary of the set $\{\rho = 1\}$. After establishing the existence of minimizers, I will characterize the ground state when the attraction is given by a quadratic interaction and the repulsion by Newtonian potential, and comment on qualitative properties of local minimizers. This is a joint work with R. Choksi.

VITALI VOUGALTER, University of Cape Town
Existence of stationary solutions for some nonlocal reaction-diffusion equations

The paper is devoted to the existence of solutions of a nonlocal reaction-diffusion equation arising in population dynamics. The proof is based on a fixed point technique. We use solvability conditions for elliptic operators in unbounded domains which do not satisfy the Fredholm property.