Meshless Methods for Partial Differential-Algebraic Equations

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Problem

- **PDAEs** occurs frequently in various applications in mathematical modeling, physical problems, multibody mechanics, spacecraft control, and incompressible fluid dynamics.
- Index analysis of the PDAEs with respect to time index, spatial index are investigated. There are few new numerical methods proposed for PDAEs.
- A big obstacle for the meshless collocation method is that the companion matrix is generally ill-conditioned, nonsymmetric and full dense matrix, which constrains the applicability of the method to solve large scale problems.
- **Multiquadric quasi-interpolation**, one of meshless methods, possesses some advantages compared with other approaches, such as less computation complexity, better shape-preserving properties.
- To circumvent the ill-conditioned companion matrices in the meshless collocation methods with RBFs and the complexity of PDAEs, this paper is devoted to the numerical solution of PDAEs using the multiquadric quasi-interpolation methods.

Problem: Consider the linear PDAEs with coefficients of the form

$$\begin{cases} A\dfrac{\partial U(x,t)}{\partial t}+B\dfrac{\partial^2 U(x,t)}{\partial x^2}+C\dfrac{\partial U(x,t)}{\partial x}+DU(x,t)=f(x,t),\ x\in(a,b), t\in(t_0,T] \\ EU(x,t)+F\dfrac{\partial U(x,t)}{\partial v}=g(x,t),\ x\in\Gamma, t\in[t_0,T], \\ U(x,t_0)=U_0(x),\ x\in(a,b) \end{cases}$$

where $A,B,C,D\in\mathbb{R}^{m\times m}, U, f:[t_0,T]\times[a,b]\to R^m$. $\frac{\partial}{\partial v}$ is the outward normal derivative, E, F are known constant matrices, $g(x,t):[t_0,T]\times[a,b]\in R^m$ and U_0 are known functions.

Here we focus our attention on the case when at least one of the matrices A and B is singular. The two special cases when $A=0$ or $B=C=0$ lead to ordinary differential equations (ODEs) or differential algebraic equations (DAEs) which are not considered here. Therefore, in this paper we assume that $A\neq 0$ and at least one of the matrices A and B is not a zero matrix.

Methods

Quasi-interpolation scheme (ICN-QIE):

- **First step:** approximate the time derivative of the partial differential operator by a forward difference using Crank-Nicolson method, i.e.,

$$\left[A+\alpha D+\alpha\left(B\frac{\partial^2}{\partial x^2}+C\frac{\partial}{\partial x}\right)\right]U^{n+1}=\left[A-\beta D+\beta\left(-B\frac{\partial^2}{\partial x^2}-C\frac{\partial}{\partial x}\right)\right]U^n+\alpha f^{n+1}(x)+\beta f^n(x),$$

where $\alpha=\Delta t\theta\ (0<\theta\le 1)$, $t_n=t_{n-1}+\Delta t$, $\beta=(1-\theta)\Delta t$ and $U^n=U(x,t_n)$, $f^n=f(x,t_n)$ with Δt is the time step size.

- **Second step:** approximate U^n by

$$U^n(x)=\begin{bmatrix}U_1^n\\U_2^n\\\vdots\\U_m^n\end{bmatrix}\cong\begin{bmatrix}(L_cU_1^n)(x)\\(L_cU_2^n)(x)\\\vdots\\(L_cU_m^n)(x)\end{bmatrix}:=(L_cU^n)(x),$$

where $(L_cU_i^n)(x)=\sum_{j=6}^{N-5}(u_{i,j}^n-P_i^n(x_j))\cdot\Psi_{i,j}(x)+P_i^n(x), 1\le i\le m$, with $u_{i,j}^n$ is the approximation of the i th component of $U(x,t)$ at point (x_j,t_n), and

$$\Psi_{i,j}(x)=\frac{\Phi_{i,j+1}(x)-\Phi_{i,j}(x)}{2(x_{j+1}-x_j)}-\frac{\Phi_{i,j}(x)-\Phi_{i,j-1}(x)}{2(x_j-x_{j-1})}, 1\le i\le m, 3\le j\le N-2,$$

$$\Phi_{i,j}(x)=\sqrt{(x-x_j)^2+c_{i,j}^2},\quad c_{i,j} \text{ is a positive constant},$$

$$P_i^n(x)=\left[a_{10,i}^n\quad a_{9,i}^n\quad\cdots\quad a_{1,i}^n\right]\left[x^9\quad x^8\quad\cdots\quad 1\right]^T, 1\le i\le m, n=1,2,\cdots$$

is a ninth-degree polynomial such that

$$\begin{aligned} &P_i^n(x_1)=u_{i,1}^n, && P_i^n(x_N)=u_{i,N}^n,\\ &P_i^n(x_2)=u_{i,2}^n, && P_i^n(x_{N-1})=u_{i,N-1}^n,\\ &P_i^n(x_3)=u_{i,3}^n, && P_i^n(x_{N-2})=u_{i,N-2}^n, 1\le i\le m.\\ &P_i^n(x_4)=u_{i,4}^n, && P_i^n(x_{N-3})=u_{i,N-3}^n,\\ &P_i^n(x_5)=u_{i,5}^n, && P_i^n(x_{N-4})=u_{i,N-4}^n. \end{aligned}$$

- **Third step:** determine $u_{i,j}^n, i=1,\cdots,m, j=1,\cdots,N,$ the collocation method is applied at every point x_j, $j=1,\cdots,N$.

Remark: When the shape parameter $c_{i,j}\equiv c$, where c is a constant, we get the **ICN-QID** method .

Numerical experiment

Example: Consider the PDAEs (1) with $a=-1, b=1$ and

$$A=\begin{pmatrix}1&0\\0&0\end{pmatrix},\ B=\begin{pmatrix}-1&0\\0&-1\end{pmatrix},\ D=\begin{pmatrix}1&1\\1&r\end{pmatrix},$$

The shape parameters for all the calculations performed in this paper are determined by trial and error, expect in ICN-QIE ($c=0.1h^{1/3}$). E is a proper identity matrix and C, F are zero matrices. The right hand side functions U_0, f, g are chosen such that the exact solution is given by

$$U(x,t)=\begin{pmatrix}U_1(x,t)\\U_2(x,t)\end{pmatrix}:=\begin{pmatrix}(x^2-1)\cos(\pi t)\\x(1-x)e^{-t}\end{pmatrix}.$$

Non-regular collocation points: (x_i,t_j) with $x_i\ (i=1,2,\cdots,41)$

$$x_i=-1,-0.9530,-0.90,-0.8530,-0.80,-0.75,-0.70,-0.64,-0.60,-0.55, -0.50,-0.46,-0.40,-0.35,-0.30,-0.25,-0.20,-0.18,-0.10,-0.05,0, 0.05,0.08,0.15,0.20,0.24,0.32,0.35,0.41,0.45,0.50,0.53,0.60,0.65, 0.70,0.7570,0.8110,0.8560,0.91,0.94,1.$$

Index-1: $r=4$.

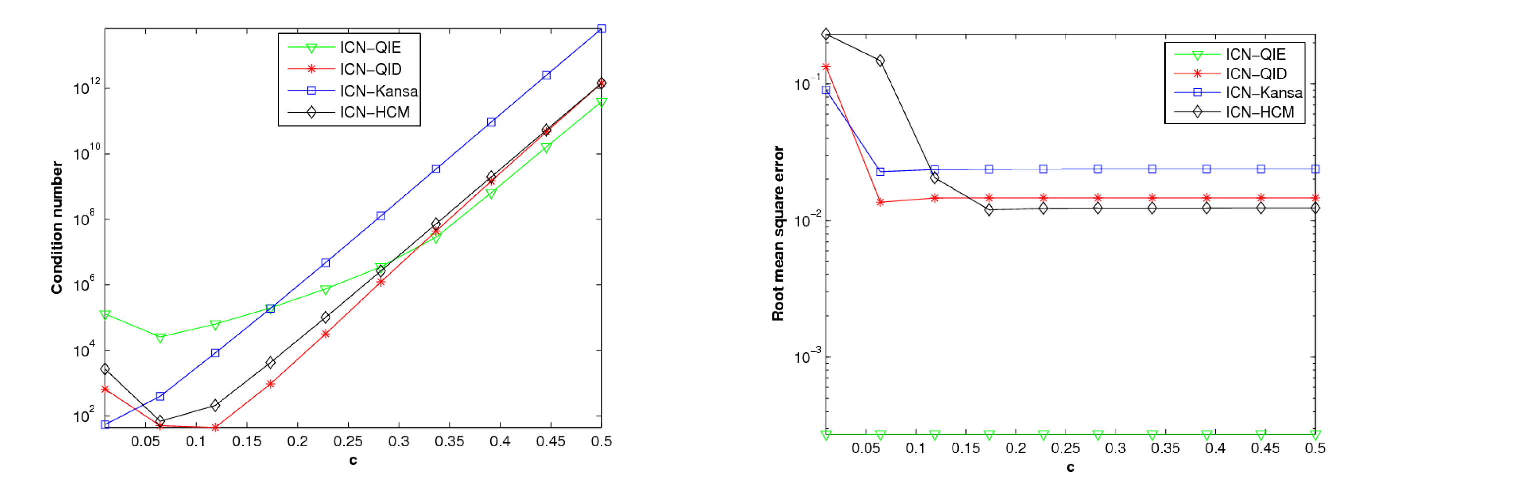

Fig. 1 Comparison of the condition numbers and the root mean square errors by the different schemes with non-regular points, where c=0.064, c=0.1733 in ICN-Kansa and ICN-CM, respectively.

Index-2: $r=-\frac{4}{h^2}\sin^2(\frac{\pi h}{4})$ and $U(x,t)=\begin{pmatrix}U_1(x,t)\\U_2(x,t)\end{pmatrix}:=\begin{pmatrix}x^5(x^2-1)\cos(\pi t)\\x^2(x^2-1)e^{-t}\end{pmatrix}$.

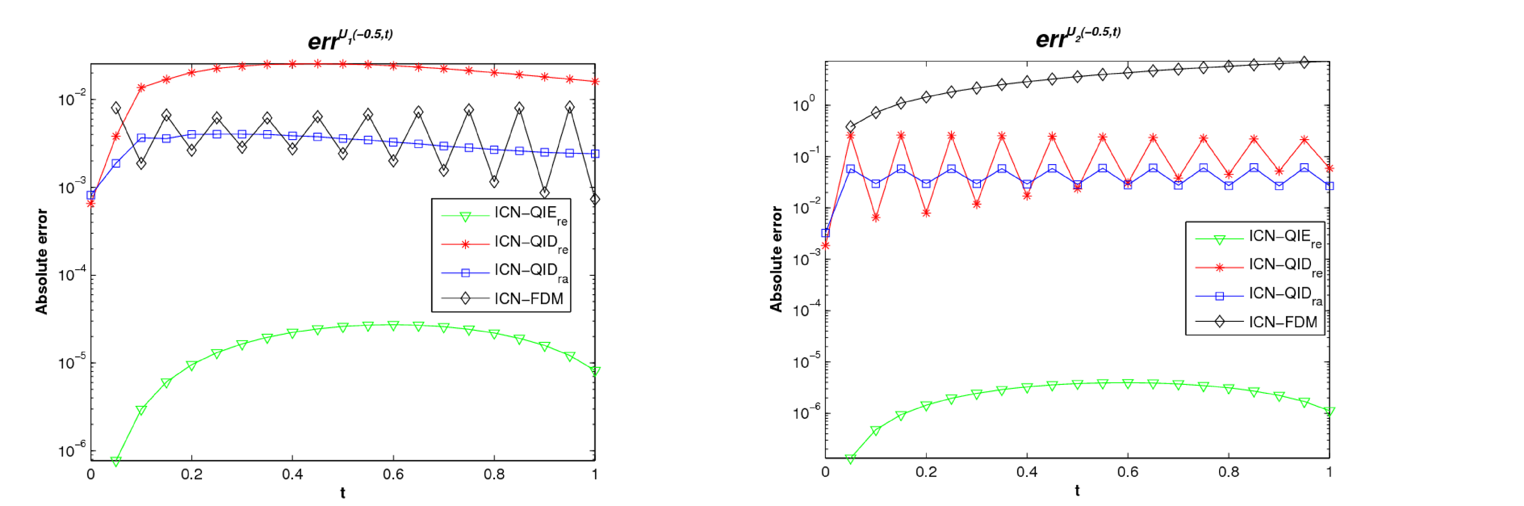

Fig. 2 Comparison of the absolute errors by the different schemes where ICN_QID_{re} (c=0.0235) and ICN_QIE_{re} are the methods with regular points, ICN_QID_{ra} (c=0.033) is with non-regular points.

Remark: *In the figures, the modal index for PDAEs is defined as by Marszalek in [1]. ICN-Kansa method refers to the implicit Crank-Nicolson with Kansa's method by multiquadrics as a RBF; ICN-HCM method refers to the implicit Crank-Nicolson with the Hermite collocation method (HCM) by multiquadrics as a RBF (for details see [2]).*

Conclusions

Conclusions:

- ICN-QIE and ICN-QID work well with non-regular collocation points, and are better than ICN-FDM for solving PDAEs with index-2 , i.e., using the randomicity of the points chosen, we have improved the numerical solutions of the PDAEs with index-2.
- By contrast with ICN-Kansa and ICN-HCM, the shape parameters of ICN-QID and ICN-QIE are easier to obtain.

Future work:

- How to deal with the non-sparse resulting matrix coefficient?
- How to choose the appropriate collocation points for PDAEs with higher index?
- How to apply the method to study the vector-borne diseases with free boundary?

References

[1] W. Marszalek, Analysis of partial differential algebraic equations, PhD Thesis, North Carolina State University, Raleigh, NC, USA, 1997.

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