An Inverse Problem for a Fractional Diffusion

Olaniyi Samuel Iyiola

Joint work with: Khaled M. Furati and Mokhtar Kirane

King Fahd University of Petroleum and Minerals, Dhahran Dammam, Saudi Arabia.

Abstract

In this paper, we present a method for determining the solution and the source term of a Riemann-Liouville time-fractional diffusion equation. Due to a nonself-adjoint boundary condition, a bi-orthogonal basis of L^2-space is selected for this purpose. The results are presented in the form of Mittag-Leffler function and an example is given to illustrate the applicability of the method.

Introduction

A special case of an inverse problem is considered for a Reimann Liouville time-fractional diffusion equation. Here, the source term, $f(x)$, is unknown and at the same time we want to determine the solution $u(x,t)$. For this purpose, we require some conditions on the boundary.

Problem Statement

We consider the problem of determining the temperature distribution, $u(x,t)$ and the source term, $f(x)$ for the following system

$$\begin{cases} D^\alpha u(x,t) - u_{xx}(x,t) = f(x),\ 0<x<1,\ 0<t<T,\ 0<\alpha\le 1, \\ I^{1-\alpha}u(x,t)|_{t=0} = g(x),\ u(x,T) = h(x),\ 0<x<1, \\ u(1,t)=0,\ u_x(0,t)=u_x(1,t),\ 0<t\le T. \end{cases}$$

where $g,\ h \in L^2(0,1)$, the initial and final conditions respectively.
The operators D^α is defined by

$$\begin{aligned} D^\alpha w(t) &= D I^{1-\alpha} w(t), \qquad D = \frac{d}{dt}, \\ I^\alpha w(t) &= \frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1} w(\tau)\, d\tau,\ t>0,\ \alpha>0, \end{aligned}$$

where Γ is the Gamma function.

Method and Construction

Using bi-orthogonal pair of dual Riesz bases for the space $L^2(0,1)$:

$$\Phi = \{\varphi_0, \varphi_{1n}, \varphi_{2n}\}_{n=1}^\infty, \quad \Psi = \{\psi_0, \psi_{1n}, \psi_{2n}\}_{n=1}^\infty,$$

where,

$$\varphi_0(x) = 2(1-x),\ \varphi_{1n}(x) = 4(1-x)\cos\lambda_n x,\ \varphi_{2n}(x) = 4\sin\lambda_n x,$$

and

$$\psi_0(x) = 1,\ \psi_{1n}(x) = \cos\lambda_n x,\ \psi_{2n}(x) = x\sin\lambda_n x,$$

with $\lambda_n = 2\pi n$.
We seek a solution and source function to our problem in the form

$$u(x,t) = u_0(t)\,\varphi_0(x) + \sum_{\substack{n=1\\k=1,2}}^\infty u_{kn}(t)\,\varphi_{kn}(x),$$

$$f(x) = f_0\,\varphi_0(x) + \sum_{\substack{n=1\\k=1,2}}^\infty f_{kn}\,\varphi_{kn}(x).$$

We can also write both initial and final data as

$$g(x) = g_0\,\varphi_0(x) + \sum_{\substack{n=1\\k=1,2}}^\infty g_{kn}\,\varphi_{kn}(x),$$

and

$$h(x) = h_0\,\varphi_0(x) + \sum_{\substack{n=1\\k=1,2}}^\infty h_{kn}\,\varphi_{kn}(x).$$

Where using the biorthogonal basis given, we have

$$g_0 = \langle g, \psi_0\rangle,\ g_{kn} = \langle g, \psi_{kn}\rangle, \qquad ,k=1,2.\ n=1,2,\cdots$$

and

$$h_0 = \langle h, \psi_0\rangle,\ h_{kn} = \langle h, \psi_{kn}\rangle, \qquad k=1,2.\ n=1,2,\cdots.$$

We denote the inner product in $L^2(0,1)$ by

$$\langle g, h\rangle = \int_0^1 g(x)\,h(x)\,dx.$$

Using these representation, we obtain the system of integro-differential equations

$$\begin{aligned} D^\alpha u_0(t) &= f_0, \\ D^\alpha u_{1n}(t) + \lambda_n^2 u_{1n}(t) &= f_{1n}, \qquad n=1,2,\ldots, \\ D^\alpha u_{2n}(t) + \lambda_n^2 u_{2n}(t) - 2\lambda_n u_{1n}(t) &= f_{2n}, \qquad n=1,2,\ldots. \end{aligned}$$

to determine f_0, f_{1n}, f_{2n}, u_0, u_{1n} and u_{2n}.
Initial conditions and final conditions are:

$$I^{1-\alpha}u_0(0) = g_0,\ \ I^{1-\alpha}u_{kn}(0) = g_{kn}, \qquad k=1,2.\ n=1,2,\cdots,$$

$$u_0(T) = h_0,\ \ u_{kn}(T) = h_{kn}, \qquad k=1,2.\ n=1,2,\cdots.$$

Results

We obtain the following results:

(1) The coefficients f_0 and f_{kn}, $k=1,2$, in the form

$$\begin{aligned} f_0 &= \frac{\Gamma(1+\alpha)}{T^\alpha}\left[h_0 - \frac{g_0}{\Gamma(\alpha)}T^{\alpha-1}\right] \\ f_{1n} &= \frac{[h_{1n} - g_{1n}T^{\alpha-1}E_{\alpha,\alpha}(-\lambda_n^2 T^\alpha)]}{T^\alpha E_{\alpha,\alpha+1}(-\lambda_n^2 T^\alpha)}, \\ f_{2n} &= \frac{[h_{2n} - g_{2n}T^{\alpha-1}E_{\alpha,\alpha}(-\lambda_n^2 T^\alpha) + S_n(T)]}{T^\alpha E_{\alpha,\alpha+1}(-\lambda_n^2 T^\alpha)}. \end{aligned}$$

(2) The coefficients u_0 and u_{kn}, $k=1,2$, in the form

$$\begin{aligned} u_0(t) &= \frac{f_0}{\Gamma(1+\alpha)}t^\alpha + \frac{g_0}{\Gamma(\alpha)}t^{\alpha-1} \\ u_{1n}(t) &= f_{1n}t^\alpha E_{\alpha,\alpha+1}(-\lambda_n^2 t) + g_{1n}t^{\alpha-1}E_{\alpha,\alpha}(-\lambda_n^2 t^\alpha), \\ u_{2n}(t) &= f_{2n}t^\alpha E_{\alpha,\alpha+1}(-\lambda_n^2 t^\alpha) + g_{2n}t^{\alpha-1}E_{\alpha,\alpha}(-\lambda_n^2 t^\alpha) + S_n(t) \end{aligned}$$

where

$$S_n(t) = 2\lambda_n\left[f_{1n}\,t^{2\alpha}\,E^2_{\alpha,2\alpha+1}(-\lambda_n^2 t^\alpha) + g_{1n}\,t^{2\alpha-1}\,E^2_{\alpha,2\alpha}(-\lambda_n^2 t^\alpha)\right].$$

Example and Numerical Results

Consider the problem with

$$g(x) = 0 \qquad \text{and} \qquad h(x) = T^\alpha(1-x).$$

Accordingly, using our results, we get

$$f(x) = \Gamma(1+\alpha)(1-x) \qquad \text{and} \qquad u(x,t) = t^\alpha(1-x).$$

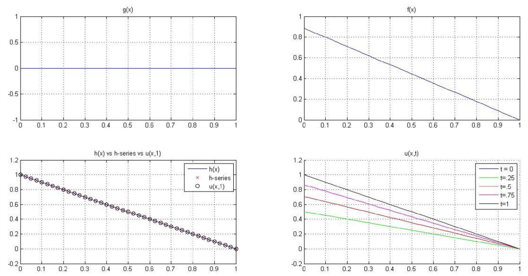

Figure 1: Solution plots with $0 < t \le 1$, $0 \le x \le 1$ and $\alpha = 0.5$

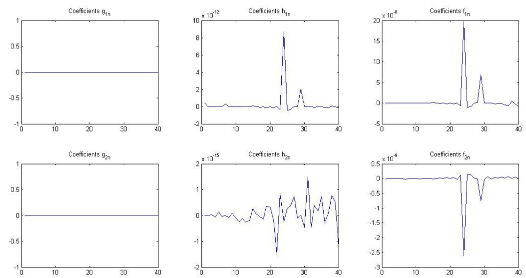

Figure 2: Convergence results for the coefficients

References

1) K.M. Furati, O.S. Iyiola and M. Kirane, An inverse problem for a generalized fractional diffusion, Applied Mathematics and Computation, 249, 24-31, 2014.
2) M. Kirane and Salman A. Malik, Determination of an unknown source term and the temperature distribution for the linear heat equation involving fractional derivative in time, Applied Mathematics and Computation, 218 (1), 163-170, 2011.

Acknowledgement

The support provided by the Canadian Mathematical Society and King Fahd University of Petroleum and Minerals is greatly acknowledged.

Contact Information

Email: samuel@kfupm.edu.sa and niyi4oau@gmail.com

Phone No.: +966-53-7611-409 and +966540475572

Presentation

The Canadian Mathematical Society (CMS) Winter Meeting at Hamilton, Ontario, Canada, December 5-8, 2014.