Abstract

Let $A \subset B$ be an extension of commutative reduced rings and $M \subset N$ an extension of positive commutative cancellative torsion-free monoids. We prove that $A$ is subintegrally closed in $B$ and $M$ is subintegrally closed in $N$ if and only if the group of invertible $A$-submodules of $B$ is isomorphic to the group of invertible $A[M]$-submodules of $B[N]$.

Assumptions

• Throughout rings are commutative and monoids are positive commutative cancellative torsion-free.
• $A \subset B$ will denote the extension of rings and $M \subset N$ will denote the extension of monoids.

Definitions

• $\mathcal{J}(A,B) :=$ The group of all invertible $A$-submodules of $B$
• The extension $A \subset A[b]$ is called elementary subintegral if $b^2, b^3 \in A$.
• The extension $A \subset B$ is called subintegral if $B = \bigcup B_i$, where each $B_i$ is obtained from $A$ by a finite succession of elementary subintegral extensions.
• The subintegral closure of $A$ in $B$, denoted by $\overline{A}$, is the largest subintegral extension of $A$ in $B$.
• We say $A$ is subintegrally closed in $B$ if $\overline{A} = A$.
• The extension $M \subset N$ is called elementary subintegral if $N = M \cup xM$ for some $x \in M$.
• Replacing $(A,B)$ by $(M,N)$ in the above, we get the similar definitions for the monoid extension.

Motivation and Introduction

The group $\mathcal{J}(A,B)$ has been studied extensively by Roberts and Singh [6]. Recently Sadhu and Singh [7], Theorem 1.5) proved that $A$ is subintegrally closed in $B$ if and only if $\mathcal{J}(A,B) \cong \mathcal{J}(A[Z],B[Z])$.

Main Theorem

(a) If $A[M]$ is subintegrally closed in $B[N]$ and $N$ is affine, then $\mathcal{J}(A,B) \cong \mathcal{J}(A[M],B[N])$.
(b) If $B$ is reduced, $A$ is subintegrally closed in $B$ and $M$ is subintegrally closed in $N$, then $\mathcal{J}(A,B) \cong \mathcal{J}(A[M],B[N])$.
(c) If $M = N$, then the reduced condition on $B$ is not needed i.e., if $A$ is subintegrally closed in $B$, then $\mathcal{J}(A,B) \cong \mathcal{J}(A[M],B[M])$.
(d) (converse of (a) and (c)) If $\mathcal{J}(A,B) \cong \mathcal{J}(A[M],B[N])$, then $A$ is subintegrally closed in $B[N]$ and $B$ is reduced or $M = N$.

Key Lemma (uses Swan-Weibel’s homotopy trick)

Let $R = R_0 \oplus R_1 \oplus \cdots$ and $S = S_0 \oplus S_1 \oplus \cdots$ be two positively graded rings with $R \subset S$ and $R_0 \subset S_0$. If the canonical map $(R,S) \to (R[X],S[X])$ is an isomorphism, then the canonical map $R(0,S_0) \to R(S_0) \to (R,S)$ is also an isomorphism.

Proof of the Key Lemma (sketch)

• $\mathcal{J}$ is a functor from the category of ring extensions to the category of abelian groups. For any morphism $\phi : (R,S) \to (R',S')$, $\mathcal{J}(\phi)$ denotes the group homomorphism $\mathcal{J}(R,S) \to \mathcal{J}(R',S')$.
• The following is a very important map. Let $w : (R,S) \to (R[X],S[X])$ be a map defined as $w(s) = s_0 + s_1X + \cdots + s_nX^n$, where $s = s_0 + s_1 + \cdots + s_n \in S$.
• Let us look at following commutative diagram where all the maps are obvious

\[
\begin{array}{ccc}
\mathcal{J}(R,S) & \xrightarrow{\mathcal{J}(w)} & \mathcal{J}(R[X],S[X]) \\
\mathcal{J}(R,S) & \xrightarrow{\mathcal{J}(w)} & \mathcal{J}(R,S) \\
\mathcal{J}(R,S) & \xrightarrow{\mathcal{J}(w)} & \mathcal{J}(R,S) \\
\end{array}
\]

• $c_1, c_2$ are evaluation map at $X = 1, X = 0$ respectively.
• Analyzing the diagram, one can conclude the Proof.

(3) Proof of the Main Theorem (a)

Since $N$ is positive affine, $N$ has a positive grading. Since $M$ is a submonoid of $N$, it has a positive grading induced from $N$.
Therefore both $A[M]$ and $B[N]$ have positive grading. Hence we can write $A[M] = A_0 \oplus A_1 \oplus \cdots$ and $B[N] = B_0 \oplus B_1 \oplus \cdots$ with $A_0 = A$, $B_0 = B$.
We define $R := A[M], S := B[N]$ and $R_0 := A_0, S_0 := B_0$. By hypothesis, $R$ is subintegrally closed in $S$, hence by Sadhu and Singh, $\mathcal{J}(R,S) \cong \mathcal{J}(R[S],S[X])$.
Therefore by the Key Lemma, we obtain that $\mathcal{J}(A,B) \cong \mathcal{J}(A[M],B[N])$.

An Interesting Corollary

Assume that $A$ is subintegrally closed in $B$ and $M$ is subintegrally closed in $N$.
(i) If $B$ is reduced or $M = N$ then $A[M]$ is subintegrally closed in $B[N]$
(ii) Conversely if $A[M]$ is subintegrally closed in $B[N]$ and $N$ is affine, then $B$ is reduced or $M = N$.

Application to Anderson’s Result

Let $A$ be a reduced seminormal ring which is Noetherian or an integral domain. Let $M$ be a positive seminormal monoid.
Let $K$ be the total quotient ring of $A$. Then $K$ is a finite product of fields, hence $Pic(K)$ is a trivial group. By Anderson [13], Corollary 2), $Pic(K[M])$ is a trivial group.
We have $U(K) = U(K[M])$ and $U(A) = U(A[M])$.

1. $U(A) \to U(K) \to Pic(A) \to Pic(K)$
2. $U(A[M]) \to U(K[M]) \to Pic(A[M]) \to Pic(K[M])$

Since by the Main Theorem $\mathcal{J}(A,K) \cong \mathcal{J}(A[M],K[M])$, we get that $Pic(A) \cong Pic(A[M])$. In this way we deduce the classical result of Anderson from the Invertible module theory.

Summary/Conclusion

• Motivated by the result $\mathcal{J}(A,B) \cong \mathcal{J}(A[X],B[X])$ of Sadhu and Singh, we proved analogous results for the positive monoids.
• It will be very interesting to see analogous result for non positive monoids.
• We have some partial results in this direction.

Remark

The results of this poster are going to appear in Journal of Commutative Algebra.

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References


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