Swan-Weibel's homotopy trick and invertible modules over monoid algebras

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Abstract

Let $A \subset B$ be an extension of commutative reduced rings and $M \subset N$ an extension of positive commutative cancellative torsion-free monoids. We prove that A is subintegrally closed in B and M is subintegrally closed in N if and only if the group of invertible A-submodules of B is isomorphic to the group of invertible $A[M]$-submodules of $B[N]$.

Assumptions

- Throughout rings are commutative and monoids are positive commutative cancellative torsion-free.
- $A \subset B$ will denote the extension of rings and $M \subset N$ will denote the extension of monoids.

Definitions

- $\mathscr{I}(A,B) :=$ The group of all invertible A-submodules of B
- The extension $A \subset A[b]$ is called **elementary subintegral** if $b^2, b^3 \in A$.
- The extension $A \subset B$ is called **subintegral** if $B = \cup_\lambda B_\lambda$, where each B_λ is obtained from A by a finite succession of elementary subintegral extensions.
- The **subintegral closure** of A in B, denoted by ${}^{+}_{B}A$, is the largest subintegral extension of A in B.
- We say A is **subintegrally closed** in B if ${}^{+}_{B}A = A$.
- The extension $M \subset N$ is called **elementary subintegral** if $N = M \cup xM$ for some x with $x^2, x^3 \in M$.
- Replacing (A,B) by (M,N) in the above, we get the similar defintions for the monoid extension.

Motivation and Introduction

The group $\mathscr{I}(A,B)$ has been studied extensively by Roberts and Singh [6]. Recently Sadhu and Singh ([7], Theorem 1.5) proved that A *is subintegrally closed in* B *if and only if* $\mathscr{I}(A,B) \cong \mathscr{I}(A[\mathbb{Z}_+], B[\mathbb{Z}_+])$.
Motivated by this result, we inquire the following statement.
A is subintegrally closed in B and M is subintegrally closed in N if and only if $\mathscr{I}(A,B)$ is isomorphic to $\mathscr{I}(A[M],B[N])$.

Main Theorem

(a) **If $A[M]$ is subintegrally closed in $B[N]$ and N is affine, then $\mathscr{I}(A,B) \cong \mathscr{I}(A[M],B[N])$.**

(b) **If B is reduced, A is subintegrally closed in B and M is subintegrally closed in N, then $\mathscr{I}(A,B) \cong \mathscr{I}(A[M],B[N])$.**

(c) **If $M = N$, then the reduced condition on B is not needed i.e. if A is subintegrally closed in B, then $\mathscr{I}(A,B) \cong \mathscr{I}(A[M],B[M])$.**

(d) **(converse of (a,b) and (c)) If $\mathscr{I}(A,B) \cong \mathscr{I}(A[M],B[N])$, then (i) $A[M]$ is subintegrally closed in $B[N]$ and (ii) B is reduced or $M = N$.**

Key Lemma (uses Swan-Weibel's homotopy trick)

Let $R = R_0 \oplus R_1 \oplus \cdots$ and $S = S_0 \oplus S_1 \oplus \cdots$ be two positively graded ring with $R \subset S$ and $R_0 \subset S_0$. If the canonical map $\theta(R,S) : \mathscr{I}(R,S) \to \mathscr{I}(R[X],S[X])$ is an isomorphism, then the canonical map $\theta(R_0,S_0) : \mathscr{I}(R_0,S_0) \to \mathscr{I}(R,S)$ is also an isomorphism.

Proof of the Key Lemma (sketch)

- $\mathscr{I}$ is a functor from the category of ring extensions to the category of abelian groups. For any morphism $\phi : (R,S) \to (R',S')$, $\mathscr{I}(\phi)$ denotes the group homomorphism from $\mathscr{I}(R,S) \to \mathscr{I}(R',S')$.
- The following is a very important map. Let $w : (R,S) \to (R[X],S[X])$ be a map defined as $w(s) = s_0 + s_1X + \cdots + s_rX^r$, where $s = s_0 + s_1 + \cdots + s_r \in S$.
- Let us look at following commutative diagram where all the maps are obvious

$$\begin{array}{ccccc}
\mathscr{I}(R,S) & \xrightarrow{\mathscr{I}(w)} & \mathscr{I}(R[X],S[X]) & \xrightarrow{\mathscr{I}(e_1)} & \mathscr{I}(R,S) \\
\downarrow{\scriptstyle \mathscr{I}(\pi)} & & \downarrow{\scriptstyle \mathscr{I}(e_0)} & & \\
\mathscr{I}(R_0,S_0) & \xrightarrow{\theta(R_0,S_0)} & \mathscr{I}(R,S). & &
\end{array}$$

- e_1, e_0 are evaluation map at $X = 1, X = 0$ respectively.
- Analyzing the diagram, one can conclude the Proof.

(3) Proof of the Main Theorem (a)

- Since N is positive affine, N has a positive grading. Since M is a submonoid of N, it has a positive grading induced from N.
- Therefore both $A[M]$ and $B[N]$ have positive grading. Hence we can write $A[M] = A_0 \oplus A_1 \oplus \cdots$ and $B[N] = B_0 \oplus B_1 \oplus \cdots$ with $A_0 = A$, $B_0 = B$.
- We define $R := A[M]$, $S := B[N]$ and $R_0 := A$, $S_0 := B$. By hypothesis, R is subintegrally closed in S, hence by Sadhu and Singh, $\mathscr{I}(R,S) \cong \mathscr{I}(R[X],S[X])$.
- Therefore by the Key Lemma, we obtain that $\mathscr{I}(A,B) \cong \mathscr{I}(A[M],B[N])$.

An Interesting Corollary

Assume that A is subintegrally closed in B and M is subintegrally closed in N.
(i) If B is reduced or $M = N$ then $A[M]$ is subintegrally closed in $B[N]$.
(ii) Conversely if $A[M]$ is subintegrally closed in $B[N]$ and N is affine, then B is reduced or $M = N$.

Application to Anderson's Result

- Let A be a reduced seminormal ring which is Noetherian or an integral domain. Let M be a positive seminormal monoid.
- Let K be the total quotient ring of A. Then K is a finite product of fields, hence $Pic(K)$ is a trivial group. By Anderson ([3], Corollary 2), $Pic(K[M])$ is a trivial qroup.
- We have $U(K) = U(K[M])$ and $U(A) = U(A[M])$.

$$\begin{array}{ccccccccc}
1 \to & U(A) & \to & U(K) & \to & \mathscr{I}(A,K) & \to & Pic(A) & \to \; Pic(K \\
& \downarrow & & \downarrow & & \downarrow & & \downarrow & \downarrow \\
1 \to & U(A[M]) & \to & U(K[M]) & \to & \mathscr{I}(A[M],K[M]) & \to & Pic(A[M]) & \to \; Pic(K[M
\end{array}$$

- Since by the Main Theorem $\mathscr{I}(A,K) \cong \mathscr{I}(A[M],K[M])$, we get that $Pic(A) \cong Pic(A[M])$. In this way we deduce the clasical result of Anderson from the Invertible module theory.

Summary/Conclusion

- Motivated by the result $\mathscr{I}(A,B) \cong \mathscr{I}(A[X],B[X])$ of Sadhu and Singh, we proved analogous results for the positive monoids.
- It will be very interesting to see analogous result for non positive monoids.
- We have some partial results in this direction.

Remark

The results of this poster are going to appear in Journal of Commutative Algebra.

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