A nonlinear generalization of the Camassa-Holm equation with peakon solutions

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Motivation

There has been much recent interest in nonlinear dispersive equations that model breaking waves. One of the first well-studied equations of this kind is the Camassa-Holm (CH) equation [3], written in the form of a system as

$$\begin{aligned} &m = u - u_{xx}, \\ &m_t = 2u_x m + u m_x = (\tfrac{1}{2}(u^2 - u_x^2) + um)_x \end{aligned} \qquad (1)$$

for $m(t,x)$, $u(t,x)$, where $u = \Delta^{-1}m$ is expressed in terms of the operator $\Delta = 1 - D_x^2$. This equation arises from the theory of shallow water waves [2, 3] and provides a model of wave breaking for a large class of solutions in which the wave slope blows up in a finite time while the wave amplitude remains bounded. There is a special class of weak solutions that describes peaked solitary waves, known as peakons [1, 3, 4].

More remarkably, the CH equation is an integrable system [3], possessing a Lax pair, a bi-Hamiltonian structure, and an infinite hierarchy of symmetries and conservation laws.

We consider a nonlinearly generalized Camassa-Holm equation, depending an arbitrary nonlinearity power $p \neq 0$. This equation reduces to the Camassa-Holm equation when $p = 1$ and shares one of the Hamiltonian structures of the Camassa-Holm equation. Two main results are obtained.

Derivation

We consider one of the Hamiltonian structures of the CH equation (1)

$$m_t = -\mathcal{E}(\delta E/\delta m) \qquad (2)$$

given by the Hamiltonian operator $\mathcal{E} = D_x - D_x^3$, where the Hamiltonian is

$$E = \int_{-\infty}^{+\infty} \tfrac{1}{2} u(u^2 + u_x^2)\, dx. \qquad (3)$$

The Hamiltonian structure (2) can be equivalently expressed in a strictly local variational form in terms of u through the identity

$$\mathcal{E}(\delta F/\delta m) = D_x(\delta F/\delta u) \qquad (4)$$

(which holds for any Hamiltonian F). This formulation gives

$$m_t = -D_x(\delta E/\delta u) \qquad (5)$$

where E is the Hamiltonian (3).

A natural nonlinear generalization of the variational formulation (5) consists of simply replacing the Hamiltonian (3) by

$$E_{(p)} = \int_{-\infty}^{+\infty} \tfrac{1}{2} u^p (u^2 + u_x^2)\, dx, \quad p \neq 0, \qquad (6)$$

which yields the Hamiltonian evolutionary equation

$$m_t = -D_x(\delta E_{(p)}/\delta u) = -\mathcal{E}(\delta E_{(p)}/\delta m), \qquad (7)$$

where p is an arbitrary nonlinearity power. The generalized CH (gCH) equation (7), written in the form of a system as

$$\begin{aligned} &m = u - u_{xx}, \\ &m_t = 2pu^{p-1}u_x m + u^p m_x + \tfrac{1}{2}p(p-1)u^{p-2}(u^2 - u_x^2)u_x \end{aligned} \qquad (8)$$

for $u(t,x)$, $m(t,x)$ reduces to the CH equation (1) when $p = 1$. For $p \neq 1$, the gCH equation (7) is a nonlinear variant of the CH equation (1).

Conservation Laws

Like the CH equation, the gCH equation has a equivalent formulation in the form of a conservation law

$$m_t = (\tfrac{1}{2}pu^{p-1}(u^2 - u_x^2) + u^p m)_x. \qquad (9)$$

Thus the integral

$$P = \int_{-\infty}^{+\infty} m\, dx \qquad (10)$$

is conserved,

$$\frac{dP}{dt} = 0 \qquad (11)$$

(under suitable asymptotic decay conditions on u). Another conserved integral is provided by the Hamiltonian (6)

$$\frac{dE_{(p)}}{dt} = 0. \qquad (12)$$

Symmetries

Proposition 1 *The infinitesimal point symmetries admitted by the gCH system (8) for $p \neq 0$ are generated by*

$$\mathbf{X}_1 = \partial_x, \quad \textit{translation in } x, \qquad (13)$$

$$\mathbf{X}_2 = \partial_t, \quad \textit{translation in } t, \qquad (14)$$

$$\mathbf{X}_3 = m\partial_m + u\partial_u - pt\partial_t, \quad \textit{scaling}. \qquad (15)$$

All of these symmetries (13)–(15) project to point symmetries of the gCH equation (9).

These symmetries are used to reduce the gCH equation to ordinary differential equations (ODEs) that describe the corresponding group invariant solutions. Reduction under the combined space-time translations (13) and (14) yields a travelling wave ODE.

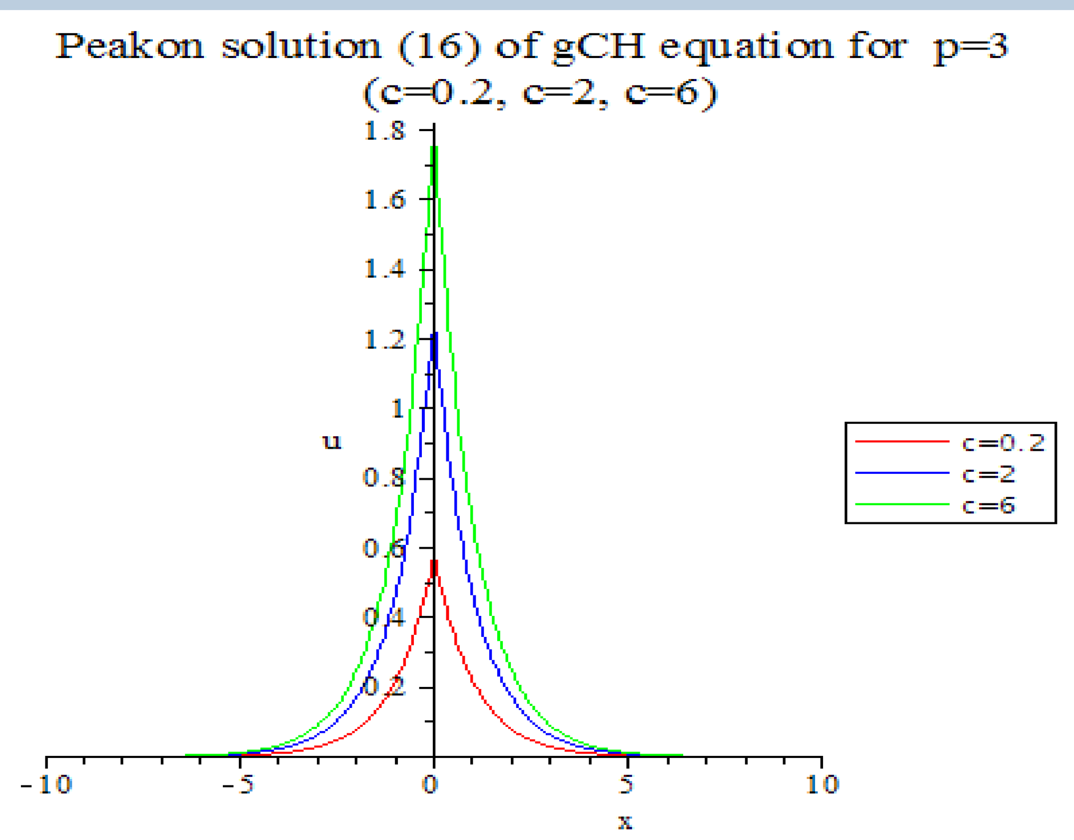
Peakon solutions

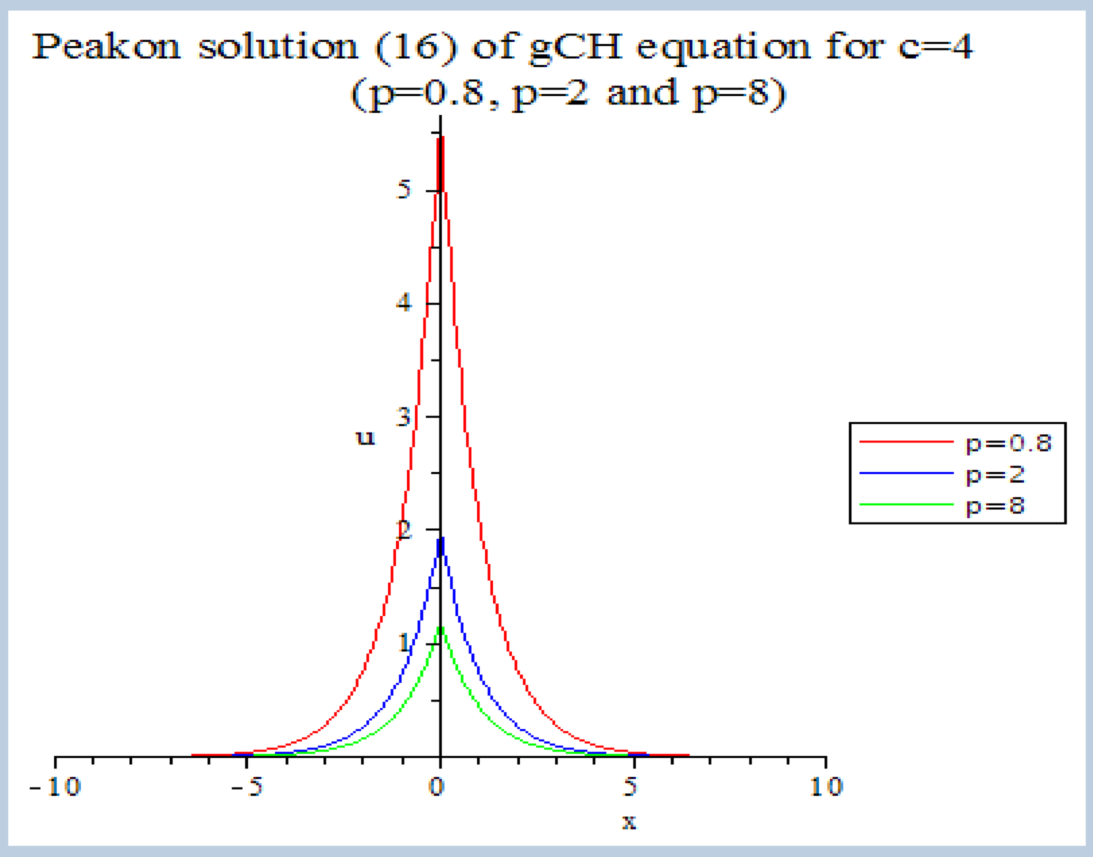
By integrating a weak form of the travelling wave ODE, peakon solutions are obtained.

Proposition 2 *The gCH equation (9) admits peaked travelling waves (which are weak solutions)*

$$u = c^{1/p}\exp(-|x + ct|), \quad p \neq 0, \qquad (16)$$

where c is an arbitrary positive constant.





Remarks

We have introduced a nonlinearly generalized CH equation (8), depending on an arbitrary nonlinearity power $p \neq 0$. This equation reduces to the CH equation when $p = 1$ and shares one of the Hamiltonian structures of CH equation (1). For all $p \neq 0$, it admits a peakon solution (16).

The gCH equation is worth further study to understand how its nonlinearity affects properties of its solutions compared to the CH equation. In particular, the CH equation is an integrable system, admits multi-peakon weak solutions, and exhibits wave-breaking for a large class of classical solutions.

- Is the gCH equation integrable for some nonlinearity power $p \neq 1$?
- Does it admit multi-peakon solutions for nonlinearity powers $p \neq 1$?
- Is it well-posed for all $p \neq 1$?
- Does it exhibit the same wave-breaking behavior for all $p \neq 1$?
- Is there a critical power p for which a different kind of blow-up occurs (other than wave breaking)?

References

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