A CONJECTURE OF WILF ON THE FROBENIUS NUMBER

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THE DIOPHANTINE FROBENIUS PROBLEM

Given coprime positive integers $a_1 < \cdots < a_d$, the **Frobenius number** is defined as

$$F = \max\left(\mathbb{Z} \setminus \left\{\sum_{i=1}^{d} \lambda_i a_i \,:\, \lambda_i \in \mathbb{N}\right\}\right).$$

The problem of determining F, known as the Diophantine Frobenius problem, is very old and has applications to several areas of pure and applied mathematics, including Commutative Algebra, Linear Algebra, Combinatorics, Coding Theory and Complexity Theory.

Example 1 (Sylvester, 1884)**.** $F(a_1, a_2) = a_1 a_2 - a_1 - a_2$.

The problem becomes considerably hard when $d \geq 3$: it is known that F cannot be expressed by algebraic formulas in terms of the a_i. Thus the research is typically focused on investigating bounds, algorithms and asymptotics.

Figure 1: The Diophantine Frobenius problem is also known as "coin problem" because of a popular formulation involving coin denominations. The so-called "McNugget numbers" represent a special case of the problem.

WILF'S CONJECTURE

Consider the following integer

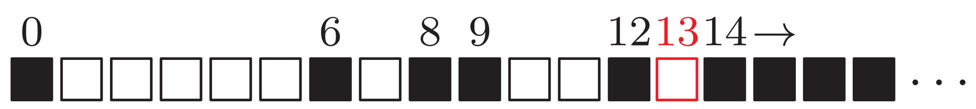
$$n = \mathrm{Card}\left([0, F] \cap \left\{\sum_{i=1}^{d} \lambda_i a_i \,:\, \lambda_i \in \mathbb{N}\right\}\right).$$

H. S. Wilf proposed an upper bound for the Frobenius number.

Conjecture 2 (Wilf, 1978)**.** The following inequality holds:

$$F + 1 \leq nd. \tag{1}$$

Example 3. Let $\{a_1, a_2, a_3, a_4\} = \{6, 8, 9, 19\}$. Denoting by ■ an integer that is representable as linear combination of the a_i with coefficients in $\mathbb{N}$, we have



Thus $d = 4$, $n = 5$, $F = 13$ and $F + 1 \leq nd$.

The conjecture has been solved in some special cases, but remains open in general.

MAIN RESULTS

Assume without loss of generality that the set $\{a_1, \ldots, a_d\}$ is irredundant, in the sense that none of the a_i is a linear combination of the remaining ones; then it is easy to check that $d \leq a_1$.
One of the first results on this problem solves the case when d is as large as possible.

Theorem 4 (Dobbs, Matthews)**.** If $d = a_1$, then Wilf's conjecture holds.

In the following theorem we verify the conjecture for the "upper half" of the possible values for d.

Theorem 5 (S.)**.** If $2d \geq a_1$, then Wilf's conjecture holds.

As a consequence, we verify the conjecture when a_1 is large enough with respect to n.

Corollary 6 (S.)**.** For each value of n Wilf's conjecture holds for all but finitely many values of a_1.

Theorems 4 and 5 concern the case when the ratio $\frac{a_1}{d}$ is bounded by 2. In the next theorem we focus on arbitray values of this ratio.

Theorem 7 (Moscariello, S.)**.** Let $\rho = \lceil \frac{a_1}{d} \rceil$. Wilf's conjecture holds if a_1 is large enough with respect to ρ and $a_1 > \rho \gcd(a_1, a_2)$.

Notice that the latter condition is satisified in particular if a_1 is not divisible by the primes smaller than ρ.

EQUALITY IN WILF'S CONJECTURE

A natural question to investigate is when does Inequality (1) become an equality. In his original paper, Wilf asked whether the equality

$$F + 1 = nd$$

is attained if and only if $d = a_1$ and $a_i = a_1 + (i - 1)$ for $i = 2, \ldots, a_1$. However, this is not the case: a simple counterexample is $a_1 = 3, a_2 = 5$. We believe that the equality can only occur in two cases:

Question 8. Does the equality $F + 1 = nd$ hold if and only if either $d = 2$ or $d = a_1$ and there exists $k \in \mathbb{N}$ such that $a_i = ka_1 + (i - 1)$ for $i = 2, \ldots, a_1$?

With the aid of `GAP`, we answer this question affirmatively when $F < n + 35$ (approximately 1.5×10^8 cases). Furthermore, we prove that Question 8 has an affirmative answer under the assumptions of Theorem 5 and Theorem 7.

ONE-DIMENSIONAL LOCAL RINGS

Let $\Bbbk$ be a field and consider the monomial curve $\mathcal{C} \subseteq \mathbb{A}^d(\Bbbk)$ defined by

$$x_1 = t^{a_1},\ x_2 = t^{a_2},\ \ldots,\ x_d = t^{a_d}.$$

$\mathcal{C}$ has a singularity at $\mathbf{0} \in \mathbb{A}^d(\Bbbk)$; in order to study it one considers the local ring $R = \Bbbk[[t^{a_1}, t^{a_2}, \ldots, t^{a_d}]]$. Let $\overline{R}$ denote the integral closure of R in $\Bbbk((t))$ and $\mathfrak{C} = (R : \overline{R})$ the conductor of R in $\overline{R}$. Denoting by $\ell(\cdot)$ the length of an R-module, the integers $\ell(R/\mathfrak{C})$ and $\ell(\overline{R}/R)$ are both measures of the singularity of R. We show that Wilf's conjecture is equivalent to the following inequality:

$$\ell(\overline{R}/R) \leq (\mathrm{edim}(R) - 1)\ell(R/\mathfrak{C}) \tag{2}$$

where $\mathrm{edim}(\cdot)$ denotes the embedding dimension. Since this version of the inequality makes sense for other classes of rings, it is natural to explore the problem in a more general context.

Question 9. Let R be a 1-dimensional Cohen-Macaulay local ring such that the integral closure $\overline{R}$ in the total ring of quotients is finite over R. Under what assumptions do we have $\ell(\overline{R}/R) \leq (\mathrm{edim}(R) - 1)\ell(R/\mathfrak{C})$?

With regards to Question 9 we prove the following result.

Theorem 10 (S.)**.** Let R be the local ring at a singular point of a reduced affine curve over an algebraically closed field. If the tangent cone of R consists of lines in uniform position then Inequality (2) holds.

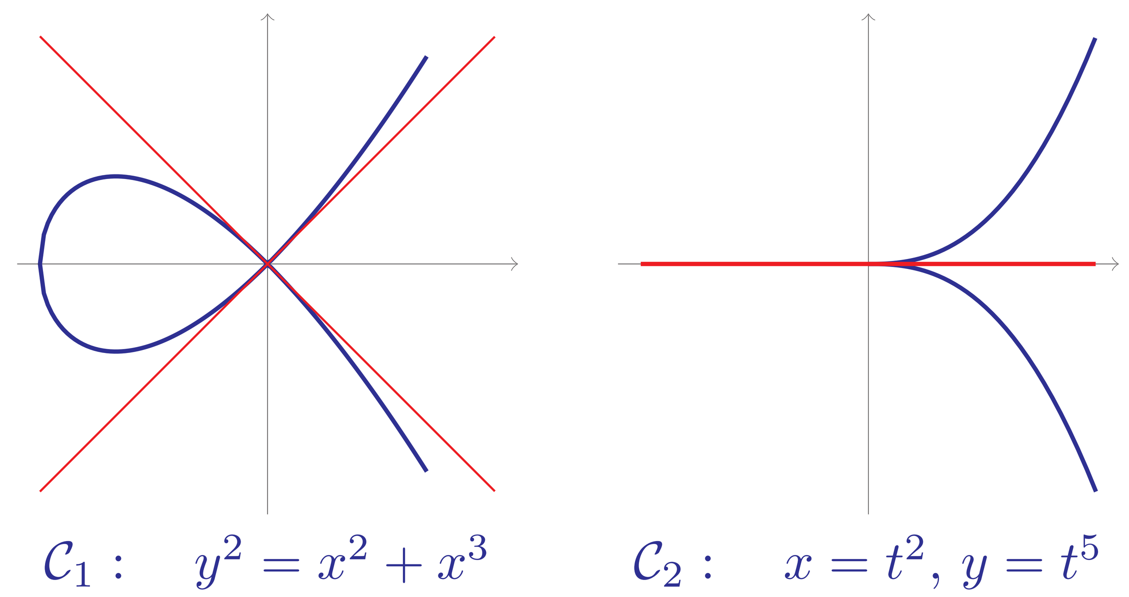


Figure 2: The tangent cone of $\mathcal{C}_1$ at $\mathbf{0}$ consists of two lines in uniform position. The (monomial) curve $\mathcal{C}_2$ has repeated tangents at $\mathbf{0}$.

REFERENCES

[1] D. E. Dobbs, G. L. Matthews, *On a question of Wilf concerning numerical semigroups*, International Journal of Commutative Rings **3**; 2003.

[2] A. Moscariello, A. Sammartano, *On a conjecture of Wilf about the Frobenius number*, Mathematische Zeitschrift (to appear); 2014. `arXiv:1408.5331`

[3] A. Sammartano, *Numerical semigroups with large embedding dimension satisfy Wilf's conjecture*, Semigroup Forum **85**, 439–447; 2012.

[4] H. S. Wilf, *A circle-of-lights algorithm for the "money-changing problem"*, American Mathematical Monthly **85**, 562–565; 1978.