Symmetries of Differential and Difference Equations
Symétries des équations différentielles et aux différences
(Org: **Alexei Cheviakov** (Saskatchewan) and/et **Pavel Winternitz** (Montréal))

STEPHEN ANCO, Brock University

Symmetry analysis and exact solutions of semilinear Schrodinger equations

A novel symmetry method is used to obtain exact solutions to Schrodinger equations with a power nonlinearity in multi-dimensions. The method uses a separation technique to solve an equivalent first-order group foliation system whose independent and dependent variables consist of the invariants and differential invariants of the point symmetry generators admitted by the Schrodinger equation. Many explicit new solutions are obtained which have interesting analytical behavior connected with blow-up and dispersion. These solutions include new similarity solutions and other new group-invariant solutions, as well as new solutions that are not invariant under any point symmetries of the Schrodinger equation. In contrast, standard symmetry reduction leads to nonlinear ODEs for which few if any explicit solutions can be derived by familiar integration methods.

ALEXANDER BIHLO, Centre de recherches mathématiques, Université de Montréal

Invariant discretization schemes

Geometric numerical integration is a recent field in the numerical analysis of differential equations. It aims at improving the quality of the numerical solution of a system of differential equations by preserving qualitative features of that system. Such qualitative feature can be conservation laws, a Hamiltonian or variational structure or a nontrivial point symmetry group. While quite some effort has been put in the construction of conservation law preserving and Hamiltonian discretization schemes, the problem of finding invariant numerical integrators is more recent and less investigated. The main obstacle one faces when constructing symmetry-preserving approximations for evolution equations is that these discretizations generally require the usage of moving meshes. Grids that undergo an evolution in the course of numerical integration pose several theoretical challenges, especially in the multi-dimensional case.

In this talk we will present three possible strategies to overcome the problem with invariant moving meshes and thus address the practicability of symmetry-preserving discretization schemes. These ways are the discretization in computational coordinates, the use of invariant interpolation schemes and the formulation of invariant meshless schemes. The different strategies will be illustrated by presenting the results obtained from invariant numerical schemes constructed for the linear heat equation, a diffusion equation and the system of shallow-water equations.

ALEXEI CHEVIAKOV, University of Saskatchewan

On Symmetry Properties of a Class of Constitutive Models in Two-dimensional Nonlinear Elastodynamics

We consider the Lagrangian formulation of the nonlinear equations governing the dynamics of isotropic homogeneous hyperelastic materials. For two-dimensional planar motions of Ciarlet–Mooney–Rivlin solids, we compute equivalence transformations that lead to a reduction of the number parameters in the constitutive law. Further, we classify point symmetries in a general dynamical setting and in traveling wave coordinates. A special value of traveling wave speed is found for which the nonlinear Ciarlet–Mooney–Rivlin equations admit an additional infinite set of point symmetries. A family of essentially two-dimensional traveling wave solutions is derived for that case.

ALFRED MICHEL GRUNDLAND, Centre de Recherches Mathematiques and Universite du Quebec a Trois-Rivieres

Soliton surfaces and zero-curvature representation of differential equations

A new version of the Fokas-Gel'fand formula for immersion of 2D surfaces in Lie algebras associated with three forms of matrix Lax pairs for either PDEs or ODEs is proposed. The Gauss-Mainardi-Codazzi equations for the surfaces are infinitesimal deformations of the zero-curvature representation for the differential equations. Such infinitesimal deformations can be constructed

from symmetries of the zero-curvature representation considered as PDE in the matrix variables or of the differential equation itself. The theory is applied to zero-curvature reprentations of the Painleve equations P1, P2 and P3. Certain geometrical aspects of surfaces associated with these Painleve equations are discussed.

Based on joint work with S. Post (University of Hawaii, USA)

VERONIQUE HUSSIN, Université de Montréal

Grassmannian sigma models and constant curvature solutions

We discuss solutions of Grassmannian models $G(m, n)$ and give some general results. We thus concentrate on such solutions with constant curvature. For holomorphic solutions, we give some conjectures for the admissible constant curvatures which are verified for the cases, $G(2, 4)$ and $G(2, 5)$. The study is extended to the case of non holomorphic solutions with constant curvatures and we show that in the case of the Veronese sequence, such curvatures are always smaller than the ones of the holomorphic solutions. This work has been done in collaboration with L. Delisle (UdM) and W. Zakrzewski (Durham, UK).

WILLARD MILLER JR., University of Minnesota

Contractions of 2D 2nd order quantum superintegrable systems and the Askey scheme for hypergeometric orthogonal polynomials

A quantum superintegrable system is an integrable n-dimensional Hamiltonian system on a Riemannian manifold with potential: $H = \Delta_n + V$ that admits 2n-1 algebraically independent partial differential operators commuting with the Hamiltonian, the maximum number possible. A system is of order L if the maximum order of the symmetry operators, other than H, is is L. For $n = 2$, $L = 2$ all systems are known. There are about 50 types but they divide into 12 equivalence classes with representatives on flat space and the 2-sphere. The symmetry operators of each system close to generate a quadratic algebra, and the irreducible representations of this algebra determine the eigenvalues of H and their multiplicity. All the 2nd order superintegrable systems are limiting cases of a single system: the generic 3-parameter potential on the 2-sphere, $S9$ in our listing. Analogously all of the quadratic symmetry algebras of these systems are contractions of $S9$. The irreducible representations of $S9$ have a realization in terms of difference operators in 1 variable. It is exactly the structure algebra of the Wilson and Racah polynomials! By contracting these representations to obtain the representations of the quadratic symmetry algebras of the other less generic superintegrable systems we obtain the full Askey scheme of orthogonal hypergeometric polynomials. This relationship provides great insight into the structure of special function theory and directly ties the structure equations to physical phenomena.

Joint work with Ernie Kalnins and Sarah Post

ROMAN POPOVYCH, Brock University

Potential symmetries in dimension three

Potential symmetries of partial differential equations with more than two independent variables are considered. Possible strategies for gauging potential are discussed. A special attention is paid to the case of three independent variables. As illustrating examples, we present gauges of potentials and nontrivial potential symmetries for the (1+2)-dimensional linear heat, Schrödinger and wave equations, the three-dimensional Laplace equation and generalizations of these equations.

SARAH POST, U. Hawaii

Contractions of superintegrable systems and limits of orthogonal polynomials

In two dimension, all second-order superintegrable systems are limits of a generic system on the sphere. These limits in the physical systems correspond to contraction of the symmetry algebras generated by the integrals of the motion as well their function space representations. The action of these limits on the representation of the models gives the well known Askey-tableau of hypergeometric polynomials.

In this talk, we focus on the top of the tableau. That is, we will discuss in depth the contractions of the generic system on the sphere to the singular isotropic oscillator of Smorodinsky and Winternitz. These limits give the limits of Wilson polynomials to Hahn, dual Hahn and Jacobi polynomials. The physical limit gives a deeper understanding of the connection between the Hahn and dual Hahn polynomials. The general theory and outline of the tableau will be discussed in a later talk of W. Miller Jr.

This is joint work with W. Miller Jr. and E. Kalnins

RAPHAËL REBELO, Université de Montréal

Symmetry preserving discretization of partial differential equations

A definition of discrete partial derivatives on non orthogonal and non uniform meshes will be given. This definition permits the application of moving frames to partial difference equations and will be used to generate invariant numerical schemes for a heat equation with source and for the spherical Burgers' equation. The numerical precision of those schemes will be displayed for two particular solutions.

DANILO RIGLIONI, CENTRE DE RECHERCHES MATHÉMATIQUES

Superintegrable systems on non Euclidean spaces

A Maximally Superintegrable (M.S.) system is an integrable n-dimensional Hamiltonian system which has 2n-1 integrals of motion. The (M.S.) systems share nice properties such as periodic trajectories for classical systems and degenerate spectrum for quantum mechanical systems. Aim of the talk is providing a complete classification of classical and quantum M.S. systems characterized by a radial symmetry and defined on n-dimensional non Euclidean manifold. We will achieve this result considering the only systems which are eligible to be M.S. namely all the classical radial systems which admit stable closed orbits and whose classification is given by the non-Euclidean generalization of the well known Bertrand's theorem. As in the Euclidean case the generalized Bertrand theorem still gives us two families of exactly solvable M.S. but, in contrast with the flat case, they exhibit extra integral of motion which have the remarkable property of being of higher order in the momenta.

ZORA THOMOVA, SUNY Institute of Technology

Contact transformations for difference equations

Contact transformations for ordinary differential equations are transformations in which the new variables $(\tilde{x}, \tilde{y})$ depend not only on the old variables (x, y) but also on the first derivative of y. The Lie algebra of contact transformations can be integrated to a Lie group. The purpose of this talk is to extend the definition of contact transformations to ordinary difference equations. We will provide an example showing that these transformations do exist. This is a joint work with D. Levi and P. Winternitz.

SASHA TURBINER, Nuclear Science Institute, UNAM

BC_2 Lame polynomials

BC_2 elliptic Hamiltonian is two-dimensional Schroedinger operator with double-periodic potential of a special form which does not admit separation of variables. In space of orbits of double-affine BC_2 Weyl group the similarity-transformed Hamiltonian takes the algebraic form of the second order differential operator with polynomial coefficients. This operator has a

finite-dimensional invariant subspace in polynomials which is a

finite-dimensional representation space of the algebra gl(3). This space is invariant wrt $2D$ projective transformations. BC_2 Lame polynomials are the eigenfunctions of this operator, supposedly, their eigenvalues define edges of the Brillouin zones (bands).

FRANCIS VALIQUETTE, Dalhousie University

Group foliation of differential equations using moving frames

We incorporate the new theory of equivariant moving frames for Lie pseudo-groups into Vessiot's method of group foliation of differential equations. The automorphic system is replaced by a set of reconstruction equations on the pseudo-group jets.

The result is a completely algorithmic and symbolic procedure for finding invariant and non-invariant solutions of differential equations admitting a symmetry group.

Joint work with Robert Thompson.

PAVEL WINTERNITZ, Universite de Montreal

Symmetry preserving discretization of ordinary differential equations

We show how one can approximate an Ordinary Differential Equation by a Difference System that has the same Lie point symmetry group as the original ODE. Such a discretization has many advantages over standard discretizations. In particular it provides numerical solutions that are qualitatively better, specially in the neighborhood of singularities.

THOMAS WOLF, Brock

ZHENGZHENG YANG, UBC