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Hamiltonian systems on Lie groups and Symmetric spaces

Lie groups G with an involutive automorphism σ provide a natural setting for variational problems that illuminate the theory of integrable Hamiltonian systems. In my talk I will attempt to justify this assertion by focusing on the following problem:

Let $\mathfrak{g}$ denote the Lie algebra of G and let $\mathfrak{g} = \mathfrak{p} + \mathfrak{k}$ denote the Cartan decomposition induced by the Lie algebra automorphism σ_*. Assume the existence of a positive definite quadratic form $\langle\ ,\ \rangle$ on $\mathfrak{k}$ and let A denote a fixed element in Cartan space $\mathfrak{p}$.

Then consider the problem of minimizing $\int_0^1 \frac{1}{2}\langle U(t), U(t)\rangle\, dt$ over the curves $g(t)$ in G that satisfy the given boundary conditions $g(0) = g_0$, $g(T) = g_1$ and are the solutions of following left invariant differential system

$$\frac{dg}{dt}(t) = g(t)\big(A + U(t)\big)\, dt$$

with $U(t)$ a bounded and measurable curve in $\mathfrak{g}$.

This "optimal control problem" admits well defined solutions for A "regular" and the Maximum Principle of optimal control induces a class of Hamiltonians H on the dual $\mathfrak{g}^*$ of $\mathfrak{g}$. The integrability of the Hamiltonian system defined by H is intimately linked to the "solvability" of the above problem.

It follows, as will be demonstrated, that H is integrable when the quadratic form $\langle\ ,\ \rangle$ is proportional to the Cartan–Killing form. In particular I will show that the most classical integrable systems, such as Jacobi's geodesic problem on the ellipsoid, Kepler's gravitational problem or the elastic problems of Kirchhoff can be seen as particular cases of the above situation.

The talk will end with some open questions.