This lecture will describe older and very recent work [2], [4] in which Bailey, Bradley and I hunted for various desired generating functions for zeta functions and then were able to methodically prove our results.

One example is

\[
3 \sum_{k=1}^{\infty} \frac{1}{\binom{2k}{k}(k^2 - x^2)} \prod_{n=1}^{k-1} \frac{4x^2 - n^2}{x^2 - n^2} = \sum_{n=1}^{\infty} \frac{1}{n^2 - x^2} \left[ = \sum_{k=0}^{\infty} \zeta(2k + 2) x^{2k} = \frac{1 - \pi x \cot(\pi x)}{2x^2} \right].
\]  

(1)

The constant term in (1) recovers the well known identity

\[
3 \sum_{k=1}^{\infty} \frac{1}{\binom{2k}{k}k^2} = \sum_{n=1}^{\infty} \frac{1}{n^2} = \zeta(2).
\]

Equivalently, for each positive integer \( k \) one has the \textit{generalized hypergeometric} identity

\[
_{3}F_{2} \left( \begin{array}{c} 3k, -k, k+1 \\ 2k+1, k+\frac{1}{2} \end{array} \right) = \frac{\binom{2k}{k}}{\binom{2k+1}{k+\frac{1}{2}}}.
\]  

(2)

As I hope to show, discovering (1) and then proving (2) formed one of the most satisfying experimental mathematics experiences I have had. I will also describe more recent work to appear in [3, 2008] regarding

\[
_{3}F_{2} \left( \begin{array}{c} 3k, -k, k+1 \\ 2k+1, k+\frac{1}{2} \end{array} \right) = \frac{\binom{2k}{k}}{\binom{2k+1}{k+\frac{1}{2}}}.
\]  

(3)

References


