Mathematical Applications of Category Theory
Applications mathématiques de la théorie des catégories

(Org: **F. William Lawvere** (SUNY Buffalo) and/et **Walter Tholen** (York))

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Categorical algebra of mapping spaces

One of the longstanding problems in homotopy theory is the question how, for a given space A, one can characterize the class of spaces which are homotopy equivalent to the pointed mapping spaces $\mathrm{Map}_*(A,Y)$. In case where A is an n-dimensional sphere S^n this problem was solved in several ways using the machinery of categorical algebra: operads, PROPs, Segal special $Delta$-spaces, etc. The common feature of all these descriptions is that they detect if a given space X is of a type of a mapping space from S^n using only certain maps between finite products of X. This shows that the mapping spaces $\mathrm{Map}_*(S^n,Y)$ are essentially algebraic objects. The talk will describe how one can try to generalize this approach to describe mapping spaces for spaces A other than S^n and the obstructions that one encounters.

This is a joint project with W. Dorabiala.

MICHAEL BARR, McGill University, Dept Math and Stats, 805 Sherbrooke W, Montreal, QC H3P 1S4

A $*$-autonomous category of topological abelian groups

Let **SPLC** denote the full subcategory of topological abelian groups consisting of the groups that can be embedded algebraically and topologically into a product of locally compact abelian groups. We show that there is a full coreflexive subcategory **C** of **SPLC** that contains all locally compact groups and is $*$-autonomous. This means that for all G, H in **C** there is an "internal hom" $G \multimap H$ whose underlying abelian group is $\mathrm{Hom}(G,H)$ and that that makes **C** into a closed category with a tensor product whose underlying abelian group is a quotient of the algebraic tensor product. Moreover a perfect duality results if we let T denote the circle group and define $G^* = G \multimap T$.

This is essentially a new exposition of work originally done jointly with H. Kleisli [Theory and Applications of Categories **8**(2001), 54–62].

JOHN BELL, Department of Philosophy, University of Western Ontario

The Axiom of Choice and the Law of Excluded Middle

In constructive mathematics the axiom of choice (AC) has a somewhat ambiguous status. On the one hand, in a topos (Diaconescu, 1975), or in intuitionistic set theory, AC entails the law of excluded middle (LEM). On the other hand, under the "propositions-as-types" interpretation which lies at the heart of constructive predicative type theories such as that of Martin-Lof, AC is actually derivable and so cannot imply LEM. One explanation for this incongruity is that the standard interpretation of AC in a topos differs from its "propositions-as-types" interpretation. Further investigation has revealed that for the derivation of LEM from AC to go through it is sufficient that sets or functions have a degree of extensionality which is built into the usual set theories but is incompatible with constructive type theories. Another condition ensuring that the derivation goes through is that any equivalence relation determines a quotient set. These facts can be given clear presentations within a weak set theory lacking the axiom of extensionality.

My talk will be devoted to these matters.

MARTA BUNGE, McGill University, Department of Mathemaics and Statistics, 805 Sherbrooke Street West, Montreal, QC H3A 2K6
Fundamental Pushout Toposes

The expression 'fundamental pushout' was employed informally in [1] in the special case of a locally connected topos $\mathcal{E}$ bounded over a base topos $\mathcal{S}$, relative to a cover U in $\mathcal{E}$. For a general topos $\mathcal{E}$ bounded over a base topos $\mathcal{S}$, we introduce here three notions which, in the locally connected case, agree. These notions, of decreasing generality, we call respectively 'locally split pushout', 'fundamental pushout' and 'Galois pushout'.

A generalization of the locally connected case is obtained in [3] by resorting to the notion of a locally constant object [4], which makes no reference to connected components. It follows that not every locally split pushout is fundamental. A condition is imposed in [3] to repair this situation but, in the process, even the locally split pushout is lost.

An alternative generalization of the locally connected case arises from the comprehensive factorization of [2]. This gives a localic and not necessarily discrete version of the fundamental pushout. This 'defect' can be corrected by imposing an assumption ('locally quasiconnected') on a Grothendieck topos $\mathcal{E}$, which states that for every object (cover) U in $\mathcal{E}$, the locale of quasicomponents of $\mathcal{E}/\mathrm{U}$ is spatial and non-trivial. The resulting fundamental pushout need not be Galois in general. In the manner of [4], one can then isolate the 'locally simply quasiconnected' toposes and derive a Galois theory.

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ROBIN COCKETT, University of Calgary
On the semantics of reversible computation

The study of reversible computation is of some foundational interest as quantum computations and fundamental physical processes are thought to be reversible. These settings, being non-classical, are particularly amenable to category theoretic techniques. The aim of the talk is to present a general structural account of reversible computation using discrete inverse categories and to state a—surprisingly general—equivalence between reversible and non-reversible computations. The importance of the equivalence is that it provides a mechanism for systematically and accurately transporting classical ideas into the reversible world.

Technically this theorem is an equivalence of categories between the category of discrete Cartesian restriction categories and the category of discrete inverse categories. Discrete inverse categories are monoidal inverse categories in which every object naturally carries a Frobenius algebra structure. The unexpected aspect—for me—was that one can uniquely reconstruct a (special sort of) "classical" world from this structure. The talk will introduce these structures and state the result!

This is joint work with Brett Giles.

JONATHON FUNK, University of the West Indies, Cave Hill Campus
The universal locally constant covering of an inverse semigroup

We examine an inverse semigroup in terms of the universal locally constant covering of its classifying topos. In particular, we prove that the fundamental group of this topos coincides with the maximum group image of the inverse semigroup. We characterize E-unitary inverse semigroups in terms of a kind of geometric morphism called a spread, characterize F-inverse semigroups, and interpret McAlister's P-theorem, which characterizes every E-unitary inverse semigroup as a kind of semidirect product of group with a poset, in terms of the universal covering morphism.

This is joint work with Ben Steinberg.

GABOR LUKACS, University of Manitoba, Winnipeg, Manitoba R3T 2N2
Categorical Methods in Topological Groups

Question Which topological groups can be embedded as closed subgroups into a product of second-countable groups?

If one omits "closed" from the question, then the answer is well known (cf. [2], [4], and [3]). Furthermore, the answer is also known to this question if "topological groups" are replaced with "Tychonoff spaces"; such spaces are called *realcompact* (cf. [1]).

Categorical methods have been successfully used in topology for nearly half a century. In this talk, the arsenal of categorical topology is used to answer this and similar questions in the area of topological groups. It turns out that the solution to the problem leads to a notion that unifies realcompactness and the Lindelöf property for topological groups.

References

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ERNIE MANES, University of Massachusetts at Amherst
Relational Models Revisited

A **flacos** is a full subcategory of the category of topological spaces and continuous maps which contains all Alexandroff spaces, and is closed under locally closed subspaces, coproducts and quotients. Such a category is topological over **Set** with products $X \otimes Y$ finer than the topological product.

Each subfunctor G of the filter functor induces a flacos $\mathbf{Top}_G$. Barr's relational models (1970) are revamped in the sense that when G is the ultrafilter functor, $\mathbf{Top}_G$ is all spaces whereas when G is the identity functor (principal ultrafilters), $\mathbf{Top}_G$ is Alexandroff spaces (= preordered sets and monotone maps). This recaptures Barr's two main examples, without using monad structure.

When G is a submonad of the ultrafilter monad, $\mathbf{Top}_G$ is a full subcategory of relational $\mathbf{G}$-models which contains all G-algebras. Such algebras X are topologically characterized by being *compact* (i.e., $X \otimes Y \to Y$ is closed for all Y) and *Hausdorff* (i.e., the diagonal is closed in $X \otimes X$). When GX is the submonad of ultrafilters which possess a countable member, $\mathbf{Top}_G$ is spaces with countable tightness and the algebras can be equationally presented using countable operations. The free countably-generated such algebra provides a counterexample to settle a question in topology posed in the 1970s.

MATIAS MENNI, Universidad Nacional de La Plata, 50 y 115, 1900 La Plata, Argentina
Abstract properties of monads arising in combinatorics

A number of examples show that certain combinatorial constructions arise as monads. Their associated Kleisli categories have a some interesting categorical properties that seem to distinguish the combinatorial context from others.

A suitable axiomatization of these properties allows to prove some facts valid for the aforementioned examples in a completely abstract way. We see this as suggesting the possibility of an abstract account of an interesting class of categories of combinatorial objects. In this talk we state the abstract properties and results and explain how they manifest in the examples.

SUSAN NIEFIELD, Union College, Department of Mathematics, Schenectady, NY 12308, USA
Lax Presheaves and Exponentiability

Replacing Set by Rel and Span in the definition of Set-valued presheaf on a category B, we study lax presheaves on B and, in each case, consider when the subcategory of exponentiable objects forms a topos.

OPEN DISCUSSION, chaired by F. William Lawvere
Trends and Problems in Category Theory

Problems that are the current preoccupaptions of category theorists, including the relationship of category theory to other fields, will be discussed by the invited speakers and other participants.

ROBERT PARE, Dalhousie University
Kan extensions for double categories

Two dimensional category theory is category theory based on Cat, the category of categories. One of the insights provided by double category theory is that Cat should be considered as a double category with functors and profunctors as arrows. Thus it is important to understand its completeness properties. Of course the whole story of limits must include Kan extensions, which are parametrized limits. We show that companions and conjoints (kinds of adjoints between horizontal and vertical arrows) are special cases of Kan extensions, and that these together with limits are sufficient for constructing Kan extensions along double functors satisfying a double Conduche condition. This is the best that can be expected as the right adjoint for "pulling back" along a functor appears as a special case.

DORETTE PRONK, Dalhousie University, Department of Math and Stats, Halifax, NS B3H 3J5
Homotopy Theory for Double Categories

In the literature there are various results relating internal double categories in the category of groups as well as certain double groups to homotopy n-types. See for example, [1], [2], and [4]. This suggests that the relationship between the homotopy theory of spaces and the homotopy theory of double categories should be investigated and clarified. Quillen model structures are particularly suited for this purpose. In this talk I will present various Quillen model structures for **DblCat**, the category of double categories, and discuss their properties.

Some of these structures are obtained by transfer along the adjunction involving the horizontal nerve, using the Lemma from Kan [3]. This lemma requires some results about certain pushouts of double categories. This led us to study a larger class of pushout diagrams of double categories and derive a nice result about the shapes of the double cells in the pushout double categories. If time permits I will discuss this result and give a sketch of the proof.

This is joint work with Simona Paoli and Tom Fiore.

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BOB ROSEBRUGH, Mount Allison University, 67 York St., Sackville, NB

Database views, lenses and monads

The 'view' concept is important for design of database systems. The view update problem: 'when can a change in a view state be extended to the total database state?' has been much studied. An early approach treated states as unstructured sets and updatable views required a 'constant complement' view, although constant factor' would have been a better name. Recently B. Pierce elucidated the older results using the notion of 'lens'. We found that lenses are algebras for a simple monad. An order-based approach with an ordered set of states provides a more appropriate approach to modelling and lenses are still algebras. When the states are the category of models for an appropriate class of sketches, a more general view of updatability emerges.

This is joint work with Michael Johnson and Richard Wood.

MYLES TIERNEY, University of Quebec at Montreal

The homotopy factorisation system of n-connected maps and n-covers

The classes of n-connected maps and n-covers form a homotopy factorisation system in the category of simplicial sets. This has important applications in the theory of Postnikov systems and elsewhere in homotopy theory, for example in the theory of n-topoi of Toen, Vezzosi and Lurie. We study some of its basic properties.

This is work with Andre Joyal.

RICHARD WOOD, Dalhousie University, Halifax, Canada

Frobenius objects in general cartesian bicategories

In recent joint work, Bob Walters and the speaker have shown that maps (left adjoint arrows) between Frobenius objects in a cartesian bicategory $\mathbf{B}$ are precisely comonoid homomorphisms and, for A Frobenius and any T in $\mathbf{B}$, $\mathrm{map}(\mathbf{B})(T, A)$ is a groupoid.

In this talk, the context of the second result will be thoroughly explained and a proof given. As a corollary, it follows that a Frobenius object in the bicategory of categories and profunctors is a groupoid. This last result was first proved about 20 years ago by Aurelio Carboni and the speaker, independently. However it is only recently that general cartesian bicategories have been defined so as to be able to contemplate the present theorem.