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Crossed Modules of Inverse Semigroups as Internal Categories

(*This is joint work with Dorette Pronk and David Zeidler.*)

A classical theorem of Spencer and Brown (1976) gives an equivalence

$$\mathsf{XMod/Group} \simeq \mathsf{Cat(Group)},$$

identifying crossed modules of groups with categories internal to Group (i.e., 2-groups). This equivalence underlies the 2-dimensional Seifert-van Kampen theorem and stands as one of the earliest uses of higher-dimensional algebra.

In this talk, we extend the Spencer-Brown equivalence to *inverse semigroups*: semigroups in which every element has a unique partial inverse, modelling local or partial symmetries. The proof of Spencer-Brown rests on a tight correspondence between split epimorphisms and semidirect products of groups, and neither construction behaves well on the nose in InvSemiGrp. After explaining how Billhardt's *full restricted semidirect product* and *split Billhardt congruences* together recover this correspondence, we introduce *Billhardt categories internal to inverse semigroups* (internal categories in InvSemiGrp whose source map is idempotent-splitting) and prove that

$$\mathsf{XMod/InvSemiGrp} \simeq \mathsf{BCat(InvSemiGrp)}.$$

The crossed module axioms must be reformulated to compensate for the failure of full invertibility, but the equivalence restricts to Spencer-Brown's on the full subcategory Group $\subset$ InvSemiGrp.