Category Theory: Structures and Applications

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FRANÇOIS BERGERON, UQAM

Functors for the working combinatorialist

As is well known, functors often describe natural constructions on the underlying objects of the source category. Paraphasing the title of the classical book by Maclane, we will illustrate how various functorial "algebras" may be turned into efficient tool to describe and specify combinatorial constructions; as well as prove natural identities between them.

DANIEL CARRANZA, Johns Hopkins University

Weak homotopy types of finite spaces

The goal of algebraic topology is to understand spaces up to weak homotopy equivalence. To that end, a natural question is to classify finite topological spaces (that is, spaces with finitely many points) up to weak homotopy equivalence. A classical result due to McCord (1966) is that, starting from a finite space, there is a convenient construction of a simplicial complex that is weakly equivalent to the space. Moreover, every finite simplicial complex is weakly equivalent to a finite space via this construction. This result often goes by the name "McCord's theorem".

In this talk, I will speak about joint work with Chris Kapulkin on a new proof of McCord's theorem, formulated in the language of abstract homotopy theory.

AMÉLIE COMTOIS, University of Ottawa

SAMUEL DESROCHERS, University of Ottawa

Extending an arithmetic universe by an object

The study of Grothendieck toposes and elementary toposes has long been a central focus of category theory. However, replacing elementary toposes by something else could offer a different approach to this theory, and using arithmetic universes – an idea of Steve Vickers – has recently found some success.

Traditionally, one studies a Grothendieck topos with respect to an elementary topos $\mathcal{S}$ called the *base*. A Grothendieck topos with respect to this base consists of another elementary topos with a bounded geometric morphism to $\mathcal{S}$. This approach has its drawbacks, though, since geometric morphisms are not the most natural notion of morphism between elementary toposes.

On the other hand, morphisms of arithmetic universes (AUs) are much closer to geometric morphisms. Vickers suggested that it might be possible to develop the theory of Grothendieck toposes using AUs. This idea has seen some success: for instance, Vickers recently developed an analogue for classifying toposes using AUs.

In this talk, I'll examine the special case of extending an AU $\mathcal{A}$ by a single indeterminate object X. I'll show that, just like for Grothendieck toposes, this yields the category $\mathrm{CoPsh}(\mathbf{Fin}_{\mathcal{A}})$ of internal copresheaves on finite sets. This is the first step in giving a more concrete description of classifying toposes over an arithmetic universe, analogous to the case of a base elementary topos.

This work is the result of PhD research under the supervision of Simon Henry and Philip J. Scott, and is based on a conjecture of Simon Henry.

ALI HAMAD, University of Ottawa

Generalised ultracategories and conceptual completeness of geometric logic

Conceptual completeness is an important result in first order categorical logic, it states that it is possible to establish a syntax semantics equivalence between a certain class of functors called ultrafunctors the category of models of a coherent theory to Set, and a pretopos (which can be thought of as a completion of the syntactic category of this theory).

Towards showing this result Makkai introduced the notion of ultracategories and ultrafunctors, these are categories equipped with an ultraproduct functor (plus data and coherence). More recently Lurie reintroduced ultracategories and showed a similar in spirit version of conceptual completeness, stating that there is an equivalence between a certain class of functors called left ultrafunctors from the category of points of a coherent topos to Set and the Topos itself.

We want to extend Makkai and Lurie's results to any topos with enough points, the first obstruction is that the category of points of such toposes do not have a canonical notion of ultraproduct. Toward this, we introduce a new notion of generalised ultracategories, where the ultraproduct may not exist, but we may find instead the "representable" at the ultraproduct. We use this new structure to show a conceptual completeness theorem for geometric logic, stating that for any two toposes with enough points E and $E^{'}$ there is an equivalence between $\mathrm{Geom}(E, E^{'})$ and $\text{Left-ultrafunctors}(\mathrm{Points}_E, \mathrm{Points}_{E'})$. This result reduces to a one similar to Lurie's one if we replace the topos $E^{'}$ by the classifying topos of the theory of objects.

SIMON HENRY, University of Ottawa
Generalized Polygraphs

Polygraphs (or equivalently computads) have been originally introduced for 2-categories by Street, and then generalized to strict infinity categories by Burroni, Street and Power, and then to many differents settings by many peoples. They are the "free models" of some given Higher algebraic theory but where "free" is taken in a very specific and iteratively defined way. And they generally appear in connection to coherence and strictification problems for these higher structures.

I will present a general notion of "polygraphs for a generalized algebraic theory" that cover all the known examples and capture this general idea of free models. At this level of generality the category of polygraphs we obtain have very poor properties, so the key point is to understand how various assumptions on the theory allows to recover various classically expected or desired properties of these category of polygraphs - I will in particular start discussing the question of when these are presheaves categories.

The long term goal is to develop a more general understanding of how properties of the category of polygraphs relates to homotopy theoretic properties of the corresponding higher structures through coherence and strictification problems, but there is still a lot of work left to get there.

(Work in progress, joint with Daniel De Almeida Souza)

BRENDA JOHNSON, Union College

CHRIS KAPULKIN, Western University
Presheaf models of dependent type theory

Here is a fundamental question in categorical logic: given a categorical model E of a logical theory and small category D, is the category $[D, \mathsf{E}]$ of E-valued presheaves on D again a model of this logical theory? Such presheaf models are ubiquitous in categorical logic; for example, in topos theory, we use the category $[[1], \mathsf{Set}]$ to show that certain toposes do not satisfy the law of excluded middle.

When studying models of dependent type theory, we have a new parameter: thanks to the work of Voevodsky, we know that each such model is canonically equipped with a class of equivalences. It thus makes sense to consider a small category D also equipped with a class of equivalences and a subcategory of the presheaf category $[D, \mathsf{E}]$ consisting of those presheaves that take equivalences to equivalences. Such constructions are again of fundamental importance, as they were used for example in the proof of Voevodsky's conjecture on homotopy canonicity of homotopy type theory.

In this talk, I will report on joint work with Fiore and Li (arXiv:2410.11728) on identifying sufficient conditions on a category

D with a class of equivalences so that $[D, \mathsf{E}]$ and its subcategory of equivalence-preserving presheaves again form a model of dependent type theory.

NATHAN KERSHAW, Western University
Categorical foundations of discrete dynamical systems

We present a categorical framework for studying discrete dynamical systems aimed at understanding modularity. The motivation for studying such systems comes partly from mathematical biology, where discrete systems such as Boolean networks, Cellular Automata, and Petri Nets are studied. These can be used to study, e.g., gene regulatory networks. One of the fundamental problems is identifying steady states in these networks. In a gene regulatory network, this could represent a stable set of gene expressions.

A discrete dynamical system is a set X with a function $f\colon X \to X$. Thus, the category of discrete dynamical systems is $\mathsf{Set}^{B\mathbb{N}}$. We can (functorially) associate to a discrete dynamical system its state space, which is a specific type of directed graph. To study state spaces, we introduce the notion of a cycle set. We can assign a cycle set to a directed graph. The composition taking a discrete dynamical system to its cycle set is a right adjoint, so in particular preserves limits. As a proof of concept of our methods, we give a conceptual proof of a generalization of a decomposition theorem of Kadelka et al. (2023).

This talk is based on joint work with D. Carranza, C. Kapulkin, R. Laubenbacher, and M. Wheeler.

ROSE KUDZMAN-BLAIS, University of Ottawa
Cartesian Linearly Distributive Categories: Revisited

Linearly distributive categories (LDC) were introduced by Cockett and Seely as alternative categorical semantics for multiplicative linear logic, taking conjunction and disjunction as primitive notions. Given that a LDC has two monoidal products, it is natural to ask when these coincide with categorical products and coproducts. Such LDCs, known as cartesian linearly distributive categories (CLDC), were introduced alongside LDCs. Initially, it was believed that CLDCs and distributive categories would coincide, but this was later found not to be the case. Consequently, the study of CLDCs was largely abandoned. In this talk, we will revisit the notion of CLDCs, demonstrating strong structural properties they all satisfy and investigating two key classes of examples: bounded distributive lattices and semi-additive categories. Additionally, we re-examine a previously assumed class of CLDCs, the Kleisli categories of exception monads of distributive categories, and show that they do not, in fact, form CLDCs.

RORY LUCYSHYN-WRIGHT, Brandon University

DIEGO MANCO, Western University

HAYATO NASU, Kyoto University, Research Institute for Mathematical Sciences
Double categories of relations relative to factorization systems and fibrations

The applications of (virtual) double categories have expanded in recent years. In this talk, we explore double categories of relations at varying levels of generality. We begin with the double category of sets, functions, and relations; we will also generalize it, first using factorization systems and then using (Grothendieck) fibrations. We will discuss the problem of characterizing those double categories arising from factorization systems and fibrations in double-categorical terms. In addition, we will show how virtual double categories naturally come into play to deal with generalized relations.

This talk is partly based on joint work with Keisuke Hoshino (Kyoto Univ) as well as my master's thesis.

BENNI NGO, University of Western Ontario
A functorial rectification of finitely cocomplete quasicategories

The study of homotopy theories can be classified into two types: the classical approach of homotopical algebra and the modern one of higher category theory.

Classical models deal with relative categories, categories equipped with a class of weak equivalences. These generalize the notion of homotopy equivalences in the category of topological spaces. This implies that constructions and invariants should be studied up to these weak equivalences instead of isomorphisms. Examples include model categories and cofibration categories.

Modern models rely on the idea that the hom-sets are replaced by spaces of morphisms. This has various implications, such as that compositions of morphisms are defined only up to contractible spaces, or that morphisms should not be compared by equality, but rather by homotopies, themselves subject to comparisons by higher homotopies. Examples include quasicategories and complete Segal spaces.

The former approach is best suited for constructing universal objects whereas the latter approach is used for working with universal properties. Therefore, one is interested in comparing these two approaches. Concretely, we would like to know how classical models translate into modern ones and vice versa.

This talk compares two models of the theory of finitely cocomplete ∞-categories: cofibration categories and finitely cocomplete quasicategories. The equivalence of their theories has originally been proved by Karol Szumiło. We will give an alternative proof that does not rely on a specific choice of a functorial localization and avoids the construction of a quasi-inverse. Instead, we exploit the finite completeness of both homotopy theories.

SUSAN NIEFIELD, Union College
Adjoints and Projectives in Double Categories of Monoids

It is well known that for a module M over a commutative ring R, the endofunctor $- \otimes_R M$ has a left adjoint if and only if M is finitely generated and projective. Following their ring/quantale analogy, Joyal and Tierney (AMS Memoirs 309, 1984) showed this characterization holds for modules over a quantale without the finiteness condition. In a paper with Wood (TAC 32, 2017), we proved a general theorem characterizing the existence of a left adjoint to $- \otimes_R M$ for modules over a monoid R in a suitable symmetric monoidal closed category $\mathcal{V}$, which we applied to obtain corollaries for rings and quantales.

Recently, Paré (Outstanding Contributions to Logic 20, Springer, 2021) considered adjoints and Cauchy completeness in double categories, and showed that an (R, Q)-bimodule M has a right adjoint in the double category of commutative rings if and only if it is finitely generated and projective as a left R-module. Subsequently (TAC 43, 2025), we incorporated this adjoint result for double categories into a version of the 2017 theorem with Wood, which we then applied to rings and quantales. However, the proofs of the latter were again separate due to the finiteness condition on rings.

In this talk, adding additional conditions on $\mathcal{V}$, we introduce a notion of $projective$ module over a monoid in $\mathcal{V}$ which includes finiteness for rings and, when added to our theorem characterizing adjoints in a double category, gives a single proof of the application to rings and quantales.

MAX PETROWITSCH, Western University
Elementary ∞-Toposes from Type Theory

Elementary toposes are categories that share many properties of the category of sets. Every elementary topos has an internal language, which is a version of typed intuitionistic higher-order logic obtained from the lattices of subobjects. The notion of elementary ∞-topos generalises this concept to ∞-categories. It is conjectured that the internal language of an elementary ∞-topos is Homotopy Type Theory (HoTT), that is Martin-Löf Dependent Type theory with Π, Σ and intensional identity types satisfying the univalence axiom. Instead of the lattice of subobjects we have a universal ∞-groupoid of all (small) objects with the structure of a space.

In the talk, I will introduce and motivate the notion of elementary ∞-topos, and I will sketch the progress that has been made so far towards proving the conjecture. I will explain how HoTT presents such an elementary ∞-topos via its syntactic category built from the syntax and rules of the type theory. First, I will use the fact that the syntactic category of HoTT has the structure of a tribe in the sense of Joyal. I will extend Joyal's theory of tribes by introducing the notion of a univalent fibration in a tribe. These fibrations exist in particular in the syntactic category of HoTT. In the second step, I will explain how each such tribe presents via its localisation an ∞-category and if the tribe has enough univalent fibrations then this ∞-category is an elementary ∞-topos.

DORETTE PRONK, Dalhousie University
Orthogonal Factorization Systems for Double Categories

In this talk we will introduce a notion of orthogonal factorization system (DOFS) for double categories that interacts well the notion of double fibration. A DOFS consists of two ordinary orthogonal factorization systems: one for the (strict) arrows of the double category and one for the double cells as arrows between the proarrows of the double category. In other words, we may think of a double category with a DOFS as a pseudo-category internal to a category of categories with an orthogonal factorization system (OFS). As categories with an OFS are algebras for a monad, there are four options for the morphisms in this category: strict, pseudo, lax and colax morphisms of algebras. (Lax morphisms of these algebras are the ones that preserve the right class of arrows.) Analogous to what was needed for double fibrations, we require that the source, target and identity morphisms are strict morphisms of algebras. There are then two versions of DOFS: the ones for which proarrow composition of double cells is a lax morphism of algebras and the ones for which it is colax.

I will discuss the details of this construction and present several examples. I will also present a 2-monad for which the double categories with a DOFS form the algebras, and describe the induced maps between these double categories. I will also discuss the interaction between these factorization systems and double fibrations.

PRIYAA SRINIVASAN, Tallinn University of Technology
Dagger-Drazin Inverses

Drazin inverses are a special kind of generalized inverses that can be defined for endomorphisms in any category. The notion of Drazin inverse can be extended to arbitrary maps (not simply endomorphisms) in the setting of dagger categories. We call this extension as dagger Drazin inverses.

In this talk, I shall introduce dagger Drazin inverses, discuss its properties and examples, and sketch out the connections between dagger Drazin inverse and its close relatives – Drazin inverse and Moore-Penrose inverse.

This is joint work with Robin Cockett and JS Pacaud Lemay.

DANIEL TEIXEIRA, Dalhousie University

WILLIAM TROIANI, University of Melbourne

JEAN-BAPTISTE VIENNEY, University of Ottawa
From tangent categories to Weil categories

Abstract: A tangent category is a category $\mathcal{C}$ equipped with an endofunctor T and some natural transformations which make T looks like the tangent bundle functor on the category smooth manifolds.

Two basic examples are the category of smooth manifolds and the category of commutative rings.

Poon Leung has proven that to make a category $\mathcal{C}$ into a tangent category is equivalent to equip it with a nice monoidal functor from a subcategory of Weil algebras generated by $\mathbb{N}[x]/(x^2)$ to the category $\mathrm{End}(\mathcal{C})$ of endofunctors of $\mathcal{C}$.

I'll explain how it could be interesting to define the notion of a Weil category as a category $\mathcal{C}$ with a nice monoidal functor from the category of all the Weil algebras to $\mathrm{End}(\mathcal{C})$.

We'll then see how both the category of smooth manifolds and the category of commutative rings should not only be tangent categories but Weil categories.

This is work in progress. The talk will explain the plan and hopefully suggest some precise definition for the notion of a Weil category.

GEOFF VOOYS, University of Calgary