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Generalised ultracategories and conceptual completeness of geometric logic

Conceptual completeness is an important result in first order categorical logic, it states that it is possible to establish a syntax semantics equivalence between a certain class of functors called ultrafunctors the category of models of a coherent theory to Set, and a pretopos (which can be thought of as a completion of the syntactic category of this theory).

Towards showing this result Makkai introduced the notion of ultracategories and ultrafunctors, these are categories equipped with an ultraproduct functor (plus data and coherence). More recently Lurie reintroduced ultracategories and showed a similar in spirit version of conceptual completeness, stating that there is an equivalence between a certain class of functors called left ultrafunctors from the category of points of a coherent topos to Set and the Topos itself.

We want to extend Makkai and Lurie's results to any topos with enough points, the first obstruction is that the category of points of such toposes do not have a canonical notion of ultraproduct. Toward this, we introduce a new notion of generalised ultracategories, where the ultraproduct may not exist, but we may find instead the "representable" at the ultraproduct. We use this new structure to show a conceptual completeness theorem for geometric logic, stating that for any two toposes with enough points E and $E^{'}$ there is an equivalence between $\mathrm{Geom}(E, E^{'})$ and $\mathrm{Left\text{-}ultrafunctors}(\mathrm{Points}_E, \mathrm{Points}_{E'})$. This result reduces to a one similar to Lurie's one if we replace the topos $E^{'}$ by the classifying topos of the theory of objects.