STEFAAN CAENEPEEL, Vrije Universiteit Brussel
Cohomology of commutative Hopf algebroids

In the first part, we discuss restricted Picard groupoid; these can be viewed as the two-dimensional version of abelian groups. Every restricted Picard groupoid is monoidal equivalent to a strict Picard groupoid, this is a Picard groupoid in which the symmetry, evaluation and coevaluation are the identity morphisms. For strict Picard groupoids, we have a structure theorem. We can define complexes of strict Picard groupoids, and the corresponding cohomology groups. To such a complex, two other complexes of abelian groups can be associated, and the cohomology groups of the three complexes are connected by a long exact sequence of cohomology, generalizing, as we will show later, the long exact sequence of Villamayor and Zelinsky. We introduce cosimplicial Picard groupoids, and associate a complex of Picard groupoids to it.

Hopf algebroids can be viewed as the proper generalization of Hopf algebras to non-commutative base rings. We will consider commutative Hopf algebroids over commutative rings; these are still more general than commutative Hopf algebras: the main difference is that we have a left and right unit map; in the case where these coincide, we recover the classical definition of commutative Hopf algebra. Other examples include: the Sweedler canonical coring associated to a commutative ring extension; $A \otimes_R H$, where H is a commutative Hopf algebra over R, and A a commutative right H-comodule algebra; this example can be generalized to the case where H is a weak Hopf algebra.

To a commutative Hopf algebroid $\mathcal{A}$, we can associate a cosimplicial commutative ring. Given a covariant functor to abelian groups or to restricted Picard groupoids, we can then construct a cosimplicial abelian group or restricted Picard groupoid. We can then consider the corresponding cohomology groups. For this covariant functor, we choose $\underline{\mathrm{Pic}}$, associating the group of invertible S-modules to a commutative ring S. The corresponding cohomology groups are then denoted $H^n(\mathcal{A}, \underline{\mathrm{Pic}})$. The above exact sequence takes the form

$$\begin{aligned} 1 \longrightarrow H^1(\mathcal{A}, \mathbb{G}_m) &\xrightarrow{\alpha_1} H^0(\mathcal{A}, \underline{\mathrm{Pic}}) \xrightarrow{\beta_1} H^0(\mathcal{A}, \mathrm{Pic}) \\ \xrightarrow{\gamma_1} H^2(\mathcal{A}, \mathbb{G}_m) &\xrightarrow{\alpha_2} H^1(\mathcal{A}, \underline{\mathrm{Pic}}) \xrightarrow{\beta_2} H^1(\mathcal{A}, \mathrm{Pic}) \\ \xrightarrow{\gamma_2} \cdots . \end{aligned}$$

In the case where $\mathcal{A} = S \otimes_R S$, the Sweedler canonical coring, the cohomology that we obtain is Amitsur cohomology, and the above exact sequence is the Villamayor–Zelinsky exact sequence; in the case where $\mathcal{A} = H$ is a commutative Hopf algebra, we recover Harrison cohomology, and, in the case where H is finitely generated and projective, Sweedler cohomology over H^*.

We give algebraic interpretations of the cohomology groups $H^n(\mathcal{A}, \underline{\mathrm{Pic}})$ in the case where $n = 0, 1, 2$. $H^0(\mathcal{A}, \underline{\mathrm{Pic}})$ is isomorphic to the Picard group of invertible $\mathcal{A}$-module corings. To describe $H^1(\mathcal{A}, \underline{\mathrm{Pic}})$, we have to describe a Galois theory of $\mathcal{A}$-module corings. In the case where $\mathcal{A} = S \otimes_R S$, the duals of the Galois coobjects are Azumaya algebras split by S. In the case where $\mathcal{A} = H$, the duals of the Galois coobjects are H-Galois objects in the classical sense. Finally, $H^2(\mathcal{A}, \underline{\mathrm{Pic}})$ classifies certain monoidal structures on the category of $\mathcal{A}$-modules. As an application of our theory, we obtain a normal basis theorem for Galois coobjects over commutative weak Hopf algebras.

Joint work with B. Femić.