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Bessel pairs and optimal Hardy and Hardy–Rellich inequalities

We give necessary and sufficient conditions on a pair of positive radial functions V and W on a ball Ω of radius R in $\mathbf{R}^n$, $n \geq 2$, so that the following inequalities hold for all $u \in C_0^\infty(\Omega)$:

$$\int_\Omega V(x)|\nabla u|^2\,dx \geq \int_\Omega W(x)u^2\,dx \tag{1}$$

and

$$\int_B V(x)|\Delta u|^2\,dx \geq \int_B W(x)|\nabla u|^2\,dx + (n-1)\int_B \Big(\frac{V(x)}{|x|^2} - \frac{v'(|x|)}{|x|}\Big)|\nabla u|^2\,dx. \tag{2}$$

We then identify a large number of such couples (V,W)—that we call Bessel pairs—and the best constants in the corresponding inequalities. This will allow us to complete, improve, extend, and unify most related results—old and new—about Hardy and Hardy–Rellich type inequalities which were obtained by Caffarelli–Kohn–Nirenberg, Brezis–Vázquez, Wang–Willem, Adimurthi–Chaudhuri–Ramaswamy, Filippas–Tertikas, Adimurthi–Grossi–Santra, as well as some very recent work by Tertikas–Zographopoulos, Liskevich–Lyachova–Moroz, and Blanchet–Bonforte–Dolbeault–Grillo–Vasquez, among others.

This is joint work with Amir Moradifam.