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Spikes for Biological Systems: The Role of Boundary Conditions

We consider the shadow system of the Gierer–Meinhardt system in a smooth bounded domain $\Omega \subset R^N$:

$$\begin{cases} A_t = \epsilon^2 \Delta A - A + \frac{A^p}{\xi^q}, & x \in \Omega,\ t > 0, \\ \tau |\Omega| \xi_t = -|\Omega| \xi + \frac{1}{\xi^s} \int_\Omega A^r \, dx, & t > 0 \end{cases}$$

with Robin boundary condition

$$\epsilon \frac{\partial A}{\partial \nu} + a_A A = 0, \quad x \in \partial \Omega,$$

where $a_A > 0$.

The positive reaction rates (p, q, r, s) satisfy

$$1 < \frac{qr}{(s+1)(p-1)} < +\infty, \quad 1 < p < \left(\frac{N+2}{N-2} \right)_+,$$

the diffusion constant is chosen such that $\epsilon \ll 1$ and the time relaxation constant such that $\tau \geq 0$.

We rigorously prove the following results on the stability of spiky solutions:

(i) If $r = 2$ and $1 < p < 1 + 4/N$ or if $r = p + 1$ and $1 < p < \infty$ then for $a_A > 1$ and τ sufficiently small the interior spike is stable.

(ii) For $N = 1$ if $r = 2$ and $1 < p \leq 3$ or if $r = p + 1$ and $1 < p < \infty$ then for $0 < a_A < 1$ the near-boundary spike is stable.

(iii) For $N = 1$ if $3 < p < 5$ and $r = 2$ then there exist $a_0 \in (0, 1)$ and $\mu_0 > 1$ such that for $a \in (a_0, 1)$ and $\mu = \frac{2q}{(s+1)(p-1)} \in (1, \mu_0)$ the near-boundary spike solution is unstable. This instability is not present for the Neumann boundary condition but only arises for Robin boundary condition. Further we show that the corresponding eigenvalue is of order $O(1)$ as $\epsilon \to 0$.

These results imply that some patterns may become more robust at the expense of others which turn unstable. Results of this type are important to understand the role of the boundary conditions in pattern selection. For some biological applications such as the modelling of skeletal limb development Robin (mixed) boundary conditions are more realistic than Neumann (zero-flux) boundary conditions which are used in most models.

This is joint work with Philip K. Maini (Oxford) and Juncheng Wei (Hong Kong).