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The double Riemann zeta function

The double Riemann zeta function $(\zeta \otimes \zeta)(s)$ is defined by a double Euler product over pairs of primes (p, q). Any nontrivial zero ρ of $(\zeta \otimes \zeta)(s)$ is given by a sum of zeros of $\zeta(s)$. Namely, there exists a pair of zeros ρ_1 and ρ_2 such that $\rho = \rho_1 + \rho_2$. The aim of this talk is to introduce a basic theory of the double Riemann zeta function, and discuss its possible applications.

The first possibility is to enlarge the zero-free region of $\zeta(s)$. We obtain an explicit form of the (p, q)-Euler factors and show that the double Euler product is absolutely convergent in $\Re(s) > 2$. Conjecturally it should be convergent in $\Re(s) > 3/2$, which implies that $\zeta(s)$ is zero-free in $\Re(s) > 3/4$. Thus any improvement of our current result would give a new result toward the RH.

The second application is to improve the ratio $N_0(T)/N(T)$. Since a zero $\rho = \rho_1 + \rho_2$ is simple only if both ρ_1 and ρ_2 are simple, an estimate of the ratio of simple zeros for $(\zeta \otimes \zeta)(s)$ can possibly improve the ratio $N_0(T)/N(T)$ for $\zeta(s)$.