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On complexity of infinite permutations

Let us say that two sequences of pairwise distinct reals $\ldots, a_1, a_2, \ldots$ and $\ldots, b_1, b_2, \ldots$ defined on the same set S (which can be finite, or equal to $\mathbb{N}$ or $\mathbb{Z}$) are equivalent if for all $i, j \in S$ we have $a_i < a_j$ if and only if $b_i < b_j$. An equivalence class of sequences on S will be called an *(S-)permutation*. An S-permutation can be also interpreted as a linear ordering of S. A permutation $\overline{a}$ having a representative $a = \ldots a_1, a_2, \ldots$ is called *t-periodic* if for all i, j such that $i, j, i+t, j+t \in S$ we have $a_i < a_j$ if and only if $a_{i+t} < a_{j+t}$. An $\mathbb{N}$-permutation is called *ultimately t-periodic* if the periodicity property holds for all $i, j \geq n_0$ for some n_0.

Surprisingly, for all $t \geq 2$ there exist infinitely many t-periodic $\mathbb{Z}$-permutations. We characterize them and give a way to code each of them.

Then we define *complexity* $f_{\overline{a}}(n)$ of a permutation $\overline{a}$ as the number of permutations (*i.e.*, equivalence classes) $\overline{a_k, a_{k+1}, \ldots, a_{k+n-1}}$. Analogously to the subword complexity of words, this function is non-decreasing, and we have:

Theorem 1 *Let $\overline{a}$ be a $\mathbb{Z}$ ($\mathbb{N}$-)permutation; then $f_{\overline{a}}(n) \leq C$ if and only if $\overline{a}$ is periodic (ultimately periodic).*

However, other properties of subword complexity cannot be directly extended to complexity of permutations: in particular, one-sided and two-sided infinite permutations have different minimal complexity.

Theorem 2 *For each unbounded growing function $g(n)$ there exists a not ultimately periodic $\mathbb{N}$-permutation $\overline{a}$ with $f_{\overline{a}}(n) \leq g(n)$ for all $n \geq n_0$. On the other hand, for each non-periodic $\mathbb{Z}$-permutation $\overline{a}$ we have $f_{\overline{a}}(n) \geq n - C$ for some constant C which can be arbitrarily large.*

This is a joint work with D. G. Fon-Der-Flaass.