RAVI VAKIL, Stanford University, Stanford, CA 94305
Murphy's Law in algebraic geometry: badly-behaved moduli spaces

We consider the question: "How bad can the deformation space of an object be?" (Alternatively: "What singularities can appear on a moduli space?") The answer seems to be: "Unless there is some *a priori* reason otherwise, the deformation space can be arbitrarily ugly." Hence many of the most important moduli spaces in algebraic geometry are arbitrarily singular, justifying a philosophy of Mumford.

More precisely, every singularity of finite type over $\mathbb{Z}$ (up to smooth parameters) appears on the Hilbert scheme of curves in projective space, and the moduli spaces of: smooth projective general-type surfaces (or higher-dimensional varieties), plane curves with nodes and cusps, stable sheaves, isolated threefold singularities, and more. The objects themselves are not pathological, and are in fact as nice as can be: the curves are smooth, the surfaces have very ample canonical bundle, the stable sheaves are torsion of rank 1, the singularities are normal and Cohen–Macaulay, *etc*.

Thus one can construct a smooth curve in projective space whose deformation space has any specified number of components, each with any specified singularity type, with any specified non-reduced behaviour along various associated subschemes. Similarly one can give a surface over $\mathbb{F}_p$ that lifts to p^7 but not p^8. (Of course the results hold in the holomorphic category as well.)