Pushing Boundaries: The Existence of Solution for a Free Boundary Problem Modeling the Spread of Ecosystem Engineers

Maryam Basiri
Supervised by: Prof. Abbas Momeni, Prof. Frithjof Lutscher
Department of Mathematics and Statistics University of Ottawa

Motivation: Spatial Spread of Species

- Many mathematical models describe spatial spread of organisms.
- The existing models assume that the habitat quality is unaffected by the presence of the species!
- But: many species modify their environment to make it more suitable [5].
- We model this process.

European Bee-Eater (M.B.)

Ecosystem Engineers

Ecosystem Engineers : species that can alter their abiotic environment and thereby enhance their population growth. Their engineering activity is essential for their survival [2].

We model the spread of Ecosystem Engineers as follows:

- Prior to engineering: unsuitable habitat.
- After engineering: suitable habitat.
- Boundary between the two types of habitat moved by the engineering activity.

North American Beaver (shutterstock.com)

Beaver Dam (oaec.org)

Beaver Dam (google.map)

Modeling Approach: Free Boundary Problem

Stefan Problem

Modeling melting of ice [8]:

- Behind the front: Water.
- Ahead of the front: Ice.
- The boundary between the two phases moves by melting.

$$\begin{cases} u_t - du_{xx} = 0, & t > 0, 0 < x < L(t), \\ u(t, L(t)) = 0 & t > 0, \\ L'(t) = -u_x(t, L(t)), & t > 0. \end{cases}$$

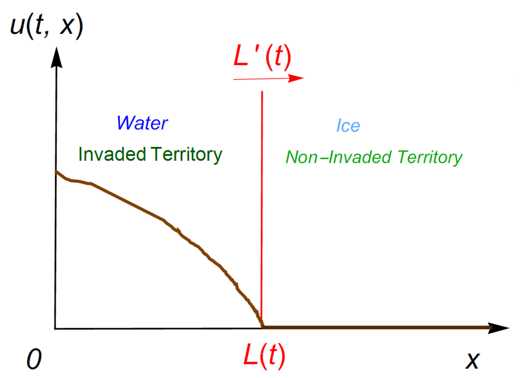

Figure: u(t,x) w.r.t x

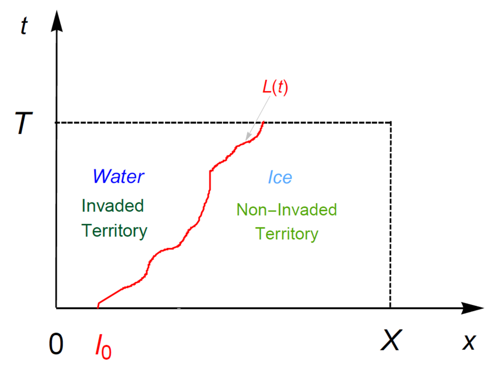

Figure: Graph of Moving boundary

Free boundary problem in Ecology

Modeling spread of invasive species [3]:

- Behind the front: The invasive species ($x < L(t)$).
- Ahead of the front: Non-invaded territory ($x > L(t)$).
- The boundary represents the spreading front ($L(t)$).

$$\begin{cases} u_t - du_{xx} = f(u), & t > 0, 0 < x < L(t), \\ u(t, L(t)) = 0, & t > 0, \\ L'(t) = -\mu u_x(t, L(t)), & t > 0. \end{cases}$$

Mathematical Model: Two-Sided Free Boundary Problems

We derive the model from an uncorrelated random walk [6]:

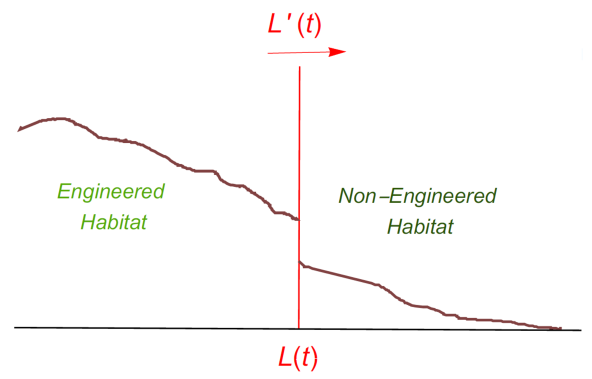

Figure: u(t,x) w.r.t x

- No clear range boundary.
- Density of the species on both side of the free boundary.
- Sharp drop in population density at the free boundary.
- Density discontinuous at the free boundary.

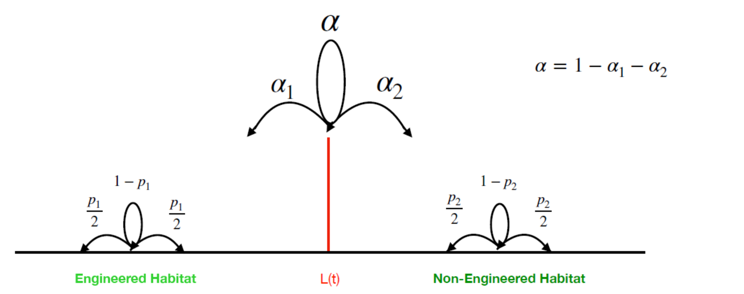

Figure: Schematic Illustration of Unbiased Random Walk:
α Probability of staying at the interface.
$\alpha_1(\alpha_2)$ Probability of jumping to (non-)engineered habitat.
$p_1(p_2)$ Probability of moving in (non-)engineered habitat.

Model Equations

$$\begin{aligned} u_t &= D_1 u_{xx} + u(r - qu) & (t, x) \in (0, T] \times (a, L(t)), \\ u_t &= D_2 u_{xx} - mu & (t, x) \in (0, T] \times (L(t), b). \end{aligned}$$

$u(x, t)$	density of the species .
$r/m > 0$	growth/death rate.
$q > 0$	intraspecific competition rate.
$D_i > 0$	diffusion rate.
$L(t)$	moving boundary.
x	space.
$t > 0$	times

Boundary Conditions

$$\begin{aligned} & u(t, L(t)^-) = ku(t, L(t)^+), & t \in (0, T], \\ & (D_1 u_x + L'(t)u) \mid_{(t, L(t)^-)} = (D_2 u_x + L'(t)u) \mid_{(t, L(t)^+)}, & t \in (0, T], \\ & u_x(t, a) = u_x(t, b) = 0, & t \in (0, T]. \end{aligned}$$

Here, $k = \frac{D_2 \alpha_1}{D_1 \alpha_2}$.
We can assume $k > 1$.

Moving Boundary Scenario

- Individuals at the boundary modify the boundary.
- The movement of the boundary is proportional to the density of individuals at the boundary.

$$L'(t) = 2D_1 \eta u(x, L(t)^-).$$

Here, $\eta > 0$ is the proportionality constant.
We can assume $L'(t) \geq 0$.

Scaled Model

To have continuity of the density at the interface $x = L(t)$, we scale the model as follows. For $t \in [0, T]$, we let

$$w(t, x) = \begin{cases} u(t, x), & x \in [a, L(t)], \\ ku(t, x), & x \in [L(t), b]. \end{cases}$$

Then, we equivalently consider the corresponding model for w: $\Rightarrow$

We need to show the existence of a curve $x = L(t)$ and a function $w = w(t, x)$ satisfying (S), (BIC) and (MC).

System (S):

$$\begin{cases} w_t = D_1 w_{xx} + w(r - qw), & (t, x) \in (0, T] \times (a, L(t)), \\ w_t = D_2 w_{xx} - mw, & (t, x) \in (0, T] \times (L(t), b). \end{cases}$$

Boundary and Initial Conditions (BIC):

For $t \in (0.T]$,

$$\begin{cases} w(t, L(t)^-) = w(t, L(t)^+), \\ (D_1 w_x + L'(t)w) \mid_{(t, L(t)^-)} = \left(D_2 \frac{w_x}{k} + L'(t) \frac{w}{k} \right) \mid_{(t, L(t)^+)}, \\ w_x(t, a) = w_x(t, b) = 0, \\ L(0) = l_0, \quad a < L(t) < b, \\ w(0, x) = w_0(x), \quad x \in [a, b]. \end{cases}$$

Moving Boundary Conditions (MC):

$$\left\{ L'(t) = 2D_1 \eta \alpha w \left(x, L(t)^- \right), \right. \qquad t \in [0, T].$$

Theorem: Existence of Solution for the Model

Suppose that $a < l_0 < b$, $w_0 \in H^1(a, b)$ with $w_0 \geq 0$ for all $x \in [a, b]$. Then there exists a solution (L, w) for the system of equations (S) with conditions (BIC) and (MC) over $[0, T]$, provided T is small [1].

Proof.

Part 1: Existence of Local Solution for a Given Curve

- We consider $T > 0$ and assume a given curve L satisfies:

$$L(0) = l_0, \quad L(t) \in (a + \delta, b - \delta), \quad L'(t) \geq 0, \quad \text{for } t \in [0, T], \quad \text{and } \delta > 0.$$

- We shall write our system in the form

$$\begin{cases} w_t(t, .) + \partial \varphi^t(w(t, .)) + \mathrm{B}(t, w(t, .)) = 0, \ 0 < t < T, \\ w(0, .) = w_0. \end{cases}$$

Define:

– The real Hilbert space $H = L^2(a, b)$ with the inner product

$$\langle u, v \rangle = \int_a^{L(t)^-} uv \, dx + \frac{1}{k} \int_{L(t)^+}^b uv \, dx, \quad \text{for all } u, v \in H \text{ and } t \in (0, T],$$

and the norm

$$\|u\|_H^2 = \|u\|^2_{L^2(a, L(t))} + \frac{1}{k} \|u\|^2_{L^2(L(t), b)}.$$

– The time-dependent functional $\varphi^t : H \to \mathbb{R} \cup \{\infty\}$

$$\varphi^t(w) := \begin{cases} \varphi_1(t, w) + \varphi_2(t, w) & \text{if } w \in H^1(a, b), \\ \infty & \text{if } w \notin H^1(a, b), \end{cases}$$

where

$$\varphi_1(t, w) = \frac{1}{2} \int_a^{L(t)} D_1 w_x^2 + w^2 \, dx + \frac{1}{2k} \int_{L(t)}^b D_2 w_x^2 + w^2 dx,$$

and

$$\varphi_2(t, w) = \frac{k}{k - 1} \frac{L'(t)}{2} \left(\frac{1}{k} w(t, L(t)^+) - w(t, L(t)^-) \right)^2.$$

– The nonlinear operator $\mathrm{B}(t, u)$,

$$\mathrm{B}(t, w) := \begin{cases} -F_1(w^+) & \text{if } x \in (a, L(t)), \\ -F_2(w^+) & \text{if } x \in (L(t), b), \end{cases}$$

with $F_1(w) = w(r - qw) + w$ and $F_2(w) = -mw + w$.

- We apply [7] to prove the existence of a solution w to the evolution equation, under appropriate assumptions on $\varphi^t : H \to [0, \infty]$ and $\mathrm{B} : H \to H$.
- We show this solution corresponds to the solution w of the system (S) with cond ition (BIC) for a given curve L.

Part 2: Existence of Local Solution for the Free boundary

- We first write the equation $L'(t) = 2D\eta\alpha w\left(x, L(t)^-\right)$ as a fixed point for an appropriate mapping [4].

For T small enough, let $\mathcal{B}$ be a closed ball in $L^2(0, T)$, with radius R. For any $r \in \mathcal{B}$ with $r(t) \geq 0$ on $[0, T]$, define:

$$\mathcal{H}(r)(t) = 2\eta D \alpha w(t, L(t)^-) \quad \text{for a.e. } t \in [0, T],$$

where $L(t) = l_0 + \int_0^t r(\tau) d\tau$ and w is the solution of the system (S) and conditions (BIC) corresponding to $L(t)$.

- We show that the operator $\mathcal{H} : \mathcal{B}^+ \to \mathcal{B}^+$ is continuous and compact.
- by Schauder's fixed point theorem, the operator $\mathcal{H}$ has a fixed point r.

Thus: the curve $L(t) = l_0 + \int_0^t r(\tau) d\tau$ and the function $w(x, t)$ are the solution to (S) with conditions (BIC) and (MC).

References

[1] Basiri et al. (2020), sub to *J. Nonlinear Sci.*
[2] Cuddington et al. (2009), *Am. Nat.*
[3] Du et al. (2010), *J. Differ. Equ.*
[4] Evans (1975), *Indiana Univ. Math.*
[5] Jones (1994), *Oikos.*
[6] Lutscher et al. (2020), *Math. Biol.*
[7] Otani (1982), *J. Differ. Equ.*
[8] Stefan (1889), *Wien Adak. Mat. Natur.*