Quantum State Transfer and Strong Cospectrality

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Abstract

A quantum spin network can be modelled by an undirected graph G whose vertices are the qubits in the network, where an edge exists between two interacting qubits. Depending on the dynamics, the evolution of the quantum spin network at time t is given by the matrix $\exp(-itH)$, where H is a Hermitian matrix associated to G. One major problem involving quantum spin networks is determining a time t such that a state of a qubit q_1 is transfered to another qubit q_2 with a particular level of probability. We call this phenomenon quantum state transfer, and the level of probability gives rise to various types of state transfer. In this presentation, we introduce different types of quantum state transfer on graphs, and present known facts. We also discuss the concept of strong cospectrality, a necessary condition for some types of quantum state transfer.

Introduction

A quantum spin system is a network of qubits that are usually interacting with each other. We model a quantum spin network using an undirected graph G, where the vertices and edges of G represent the qubits and interactions amongst qubits, respectively. A great deal of interest in quantum state transfer research focuses on the time that it takes to transmit the state of a qubit from one vertex to another given a particular level of probability. This probability is called the fidelity of quantum state transfer. Here, transmission occurs by initializing the state at one vertex and wait for the state to naturally propagate to the other vertex.

There are two common types of dynamics governing quantum state transfer, namely the XY (or adjacency) dynamics and the Heisenberg (or Laplacian) dynamics. The Hamiltonian of the quantum system is represented by a Hermitian matrix H. For XY dynamics, H is the adjacency matrix of the underlying graph, and for the Heisenberg dynamics, H is the Laplacian matrix of the underlying graph. It is known that the evolution of the quantum spin network at time t is given by the transition matrix $U(t) = e^{-itH}$, which is a complex symmetric unitary matrix. If $U(t) = [u(t)_{ij}]$, then $|u(t)_{ij}|^2$ gives the fidelity of state transfer from vertex i to vertex j at time t.

Preliminaries

Definition 1. Let G be a graph with distinct vertices i and j.

1. We say that G admits *(adjacency or Laplacian) perfect state transfer* (PST) from vertex i to vertex j if there exists a time τ and some unit complex number γ such that
$$U(\tau)e_i = \gamma e_j.$$
Equivalently, $|u(\tau)_{ij}| = 1$.
2. We say that G is *(adjacency or Laplacian) periodic* at vertex i if there exists a time τ such that $|u(\tau)_{ii}| = 1$.
3. We say that G admits *(adjacency or Laplacian) pretty good state transfer* (PGST) from vertex i to vertex j if there exists a sequence of times $\{\tau_k\}$ and a unit complex number γ such that
$$\lim_{k\to\infty} ||U(\tau_k)e_i - \gamma e_j|| = 0.$$
Equivalently, $|u(\tau_k)_{ij}| \to 1$ as $k \to \infty$.
4. We say that G admits *(adjacency or Laplacian)* (α, β)*-fractional revival* (FR) from i to j if there exists a time τ and complex numbers α, β such that $|\alpha|^2 + |\beta|^2 = 1$ such that
$$U(\tau)e_i = \alpha e_i + \beta e_j$$
We say that the (α, β)-fractional revival from u to v at time τ is *proper* if $\alpha, \beta \neq 0$ and *balanced* if $|\alpha| = |\beta| = \frac{1}{\sqrt{2}}$.
5. We say that G admits *(adjacency or Laplacian) (instantaneous) uniform mixing* at time τ if all entries of $U(t)$ have equal absolute values.

Remarks.

1. PST is symmetric and monogamous [2].
2. If there is PST from i to j at time t, then u and v are periodic at time $2t$. The converse is not true.
3. If G admits (instantaneous) uniform mixing at time τ, then $|u(\tau)_{ij}| = \frac{1}{\sqrt{n}}$ for all i, j.
4. Fractional revival is weakly symmetric [1].
5. If G is a regular graph, then the transition matrices with respect to the XY and Heisenberg dynamics are equivalent.

Theorem 2. (Spectral Decomposition) Let H be a Hermitian matrix. Then we can write $H = \sum_{i=1}^{r} \lambda_i E_i$, where $\lambda_1, \ldots, \lambda_r$ are the distinct eigenvalues of H and $E_1, \ldots, E_r$ are the orthogonal projection matrices onto the eigenspaces corresponding to the λ_i's.

Note that the adjacency A and Laplacian L matrices of a graph are Hermitian matrices.

Definition 3. Let X be a graph with distinct vertices i and j.

1. We say that i and j are *cospectral* if $(E_k)_{ii} = \pm(E_k)_{jj}$ for all k, *parallel* if for all k, there exists a constant c such that $E_k e_i = cE_k e_j$, and *strongly cospectral* if $E_k e_i = \pm E_k e_j$ for all k.
2. The *eigenvalue support* σ_i of vertex i is the set of all eigenvalues λ_k such that $E_k e_i \neq 0$.

Note that the eigenvalue supports of two strongly cospectral vertices are equal. Moreover, it is known that two vertices i and j in a graph G are strongly cospectral if and only if they are cospectral and parallel.

Example

Consider the complete graph on two vertices K_2 with adjacency matrix $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. Then $A^{2k} = I$ and $A^{2k} = A$ for all integers k. Using the power series expansion of e^x, $\cos x$ and $\sin x$, we obtain
$$e^{-itA} = \sum_{k=0}^{\infty} \frac{(-itA)^k}{k!} = \sum_{k=0}^{\infty} \frac{(-itA)^{2k}}{(2k)!} + \sum_{k=0}^{\infty} \frac{(-itA)^{2k+1}}{(2k+1)!} = (\cos t)I + i(\sin t)A.$$
Therefore, $U(t) = \begin{bmatrix} \cos t & i \sin t \\ i \sin t & \cos t \end{bmatrix}$. We have the following observations.

1. Since $|u\left(\frac{\pi}{2}\right)_{12}| = 1$, K_2 admits PST between vertices 1 and 2 at time $\frac{\pi}{2}$.
2. Since $|u(\pi)_{ii}| = 1$ for $i = 1, 2$, K_2 is periodic at vertices 1 and 2 at time π.
3. Since $u\left(\frac{\pi}{4}\right)_{ij}$ are equal for all i, j, K_2 admits (instantaneous) uniform mixing.

Known Results

Periodicity

Let X be a graph with vertex i. The *ratio condition* holds at i if for all $\theta_p, \theta_r, \theta_s, \theta_t \in \sigma_u$ with $\theta_s \neq \theta_t$,
$$\frac{\theta_p - \theta_r}{\theta_s - \theta_t} \in \mathbb{Q}$$

Theorem 4. [3] A graph G is (adjacency or Laplacian) periodic at vertex i if and only if the ratio condition holds at i.

Theorem 5. [3] A graph G is (adjacency or Laplacian) periodic at vertex i if and only if either

1. all eigenvalues in σ_i are integers; or
2. all eigenvalues in σ_i are quadratic integers in $\mathbb{Q}(\Delta)$ for some square-free integer Δ such that the difference of any two eigenvalues in σ_i is an integer multiple of Δ.

We say that the graph G is *(adjacency or Laplacian) almost periodic* if there exists a sequence of times $\{\tau_k\}$ and a unit complex number γ such that
$$\lim_{k\to\infty} ||U(\tau_k)e_i - \gamma I|| = 0.$$

Theorem 6. [5] Every graph G is almost periodic.

Corollary 7. Every vertex i in a graph G is almost periodic.

Example 8. For any $n \geq 3$, the complete graph K_n is periodic at every vertex at time $t = \frac{\pi}{n}$.

Perfect State Transfer

Theorem 9. [6] Let G be a graph with vertices i and j. Let $\lambda_0, \ldots, \lambda_k$ be the eigenvalues in σ_i. Then G admits (adjacency or Laplacian) PST between i and j if and only if the following hold.

1. Vertices i and j are strongly cospectral.
2. Nonzero elements in σ_i are either all integers or all quadratic integers. Moreover, there is a square-free integer Δ, an integer a and integers $b_0, \ldots, b_k$ such that $\lambda_r = \frac{1}{2}(a + b_r\sqrt{\Delta})$ for all $r = 0, 1, \ldots, k$.
3. Let $g_k = \frac{\lambda_0 - \lambda_r}{\sqrt{\Delta}}$ and $g = \gcd\{g_k\}$. Then $E_k e_i = E_k e_v$ if and only if $\frac{g_k}{g}$ is even, and $E_k e_i = -E_k e_v$ if and only if $\frac{g_k}{g}$ is odd.

Theorem 10. [4] There are only finitely many connected graphs with maximum valency at most k where perfect state transfer occurs.

Example 11. The complete graph K_n does not exhibit PST for any $n \geq 3$.

Pretty Good State Transfer

For two strongly cospectral vertices i and j of a graph G, define $\gamma_k = \begin{cases} 1, & \text{if } E_k e_i = -E_k e_j \\ 0, & \text{if } E_k e_i = E_k e_j. \end{cases}$

Theorem 12. [5] Let G be a graph with vertices i and j. Let $\lambda_0, \ldots, \lambda_k$ be the eigenvalues in σ_i. Then G admits (adjacency or Laplacian) PGST between i and j if and only if the following hold.

1. Vertices i and j are strongly cospectral.
2. If there is a set of integers $\{\ell_k\}$ such that $\sum_{\lambda_k \in \sigma_i} \ell_k \lambda_k = 0$ and $\sum_{\lambda_k \in \sigma_i} \ell_k \gamma_k$ is odd, then $\sum_{\lambda_k \in \sigma_i} \ell_k \neq 0$.

The proof of the theorem above utilizes Kronecker's Theorem.

Example 13. The path P_n exhibits adjacency PST only when $n = 2, 3$ and Laplacian PST only when $n = 2$. However, using the theorem above, it is shown in [5] that P_n exhibits adjacency and Laplacian PGST for infinitely many values of n.

Fractional Revival

Proposition 14. [1] If there is adjacency (α, β)-fractional revival between vertices i and j in a graph X, then these vertices are parallel.

Proposition 15. [1] Let X be a graph that admits adjacency (α, β)-fractional revival between vertices i and j at time τ. Then i and j are strongly cospectral if and only if there exists $\gamma, \zeta \in \mathbb{C}$ such that $\alpha = e^{i\zeta} \cos\gamma$ and $\beta = ie^{i\zeta} \sin\gamma$.

Theorem 16. [1] Let G be a graph with strongly cospectral vertices i and j. Then adjacency $(e^{i\zeta} \cos\gamma, ie^{i\zeta} \sin\gamma)$-fractional revival occurs between i and j at time τ if and only if for all $\lambda_k \in \sigma_i$,
$$\tau(\lambda_0 - \lambda_k) = \begin{cases} 0 \pmod{2\pi}, & \text{if } E_k e_i = E_k e_j \\ -2\gamma \pmod{2\pi}, & \text{if } E_k e_i = -E_k e_j. \end{cases}$$

Theorem 17. [1] If fractional revival occurs between two strongly cospectral vertices i and j in G, then G has PST or PGST from i to j, or G is periodic at i and j at the same time.

Theorem 18. [1] There are only finitely many connected graphs with maximum valency at most k where adjacency fractional revival between cospectral vertices occur.

Example 19. A path P_n admits adjacency fractional revival if and only if $n \in \{2, 3, 4\}$.

Future Work

Fractional revival largely remains unexplored, especially Laplacian fractional revival. One can also consider generalizations of fractional revival. To generate more families of graphs that exhibit PST, PGST and fractional revival, one can also look at which graph operations induce strong cospectrality.

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