The Radius of the Comparison of the Crossed Product by a Tracially Strictly Approximately Inner Action

M. Ali Asadi-Vasfi

University of Tehran

2020 CMS Winter Meeting

Introduction

The radius of comparison is a numerical invariant, based on the Cuntz semigroup, which was introduced by Andrew Toms in [3]. The radius of comparison plays a key role in the Elliott program for the classification of C*-algebras.

Roughly speaking, among simple nuclear C*-algebras, the classifiable ones are those whose radii of comparison are zero and the nonclassifiable ones are those whose radii of comparison are strictly positive.

With the near completion of the Elliott program, nonclassifiable C*-algebras attract more attention. Part of my Ph.D. project is to conduct an investigation beyond classifiable C*-algebras by considering crossed products by finite groups of simple C*-algebras.

Problem Let A be a unital C*-algebra, let G be a finite group, and let $\alpha\colon G \to \mathrm{Aut}(A)$ be an action of G on A. How are the radius of comparison of the crossed product $A \rtimes_\alpha G$ and of the fixed point algebra A^α related to the radius of comparison of A?

Setting

Definition Let A be a C*-algebara. Let $m, n \in \mathbb{Z}_{>0}$, let $a \in M_n(A)_+$, and let $b \in M_m(A)_+$. We say that a is *Cuntz subequivalent* to b in A, written $a \precsim_A b$, if there exists a sequence $(x_k)_{k=1}^\infty$ in $M_{n,m}(A)$ such that

$$\lim_{k\to\infty} x_k b x_k^* = a.$$

We say that a is *Cuntz equivalent* to b in A, written $a \sim_A b$, if $a \precsim_A b$ and $b \precsim_A a$.

Definition Let A be a stably finite unital C*-algebra. We let $\mathrm{QT}(A)$ be the set of all normalized quasitraces on A (the quasitrace of the identity of $M_n(A)$ is n, not 1).

- For every $\tau \in \mathrm{QT}(A)$ and $a \in \bigcup_{k=1}^\infty M_k(A)_+$, define $d_\tau(a) = \lim_{n\to\infty} \tau(a^{\frac{1}{n}})$.
- For $r \in [0, \infty)$, A has *r-comparison* if whenever $a, b \in \bigcup_{k=1}^\infty M_k(A)_+$ satisfy $d_\tau(a) + r < d_\tau(b)$ for all $\tau \in \mathrm{QT}(A)$, then $a \precsim_A b$.
- The *radius of comparison* of A, denoted $\mathrm{rc}(A)$, is

$$\mathrm{rc}(A) = \inf\left(\left\{ r \in [0,\infty) \colon A \text{ has } r\text{-comparison} \right\}\right).$$

We take $\mathrm{rc}(A) = \infty$ if there is no r such that A has r-comparison.

Tracial Strict Approximate Innerness

Definition Let A be an infinite-dimensional simple unital C*-algebra, let G be a finite group, and let $\alpha\colon G \to \mathrm{Aut}(A)$ be an action of G on A. We say that α is *tracially strictly approximately inner* if for every finite set $F \subset A$, every $\varepsilon > 0$, and every positive element $x \in A$ with $\|x\| = 1$, there are a projection $e \in A$ and unitaries $z_g \in \mathrm{U}(eAe)$ for $g \in G$ such that:

- $\|ea - ae\| < \varepsilon$ for all $a \in F$.
- $\|\alpha_g(eae) - z_g eae z_g^*\| < \varepsilon$ for all $a \in F$ and all $g \in G$.
- $\|z_g z_h - z_{gh}\| < \varepsilon$ for all $g, h \in G$.
- $1 - e \precsim_A x$.
- $\|exe\| > 1 - \varepsilon$.

Motivating Example

Example Let A be a stably finite unital C*-algebra and take G to be $\mathbb{Z}/2\mathbb{Z}$

- If $\alpha\colon G \to \mathrm{Aut}(A)$ is the trivial action, then

$$A \rtimes_\alpha G \cong A \oplus A \quad \text{and} \quad \mathrm{rc}\left(A \rtimes_\alpha G\right) = \mathrm{rc}(A \oplus A) = \mathrm{rc}(A).$$

- If $\beta\colon G \to \mathrm{Aut}(A \oplus A)$ is the flip action, then

$$A \rtimes_\beta G \cong M_2(A) \quad \text{and} \quad \mathrm{rc}\left(A \rtimes_\beta G\right) = \mathrm{rc}(M_2(A)) = \tfrac{1}{2} \cdot \mathrm{rc}(A).$$

This example suggests that $\mathrm{rc}(A \rtimes_\alpha G) = \frac{1}{\mathrm{card}(G)} \cdot \mathrm{rc}(A)$ for sufficiently outer actions and $\mathrm{rc}(A \rtimes_\alpha G) = \mathrm{rc}(A)$ for sufficiently inner actions.

Main Results

Proposition Let A be an infinite-dimensional simple unital C*-algebra, let $a, b \in A_+$, and let $\alpha\colon G \to \mathrm{Aut}(A)$ be an action of a finite group G on A which is tracially strictly approximately inner. Assume 0 is a limit point of $\mathrm{sp}(b)$. Then

$$a \precsim_{A \rtimes_\alpha G} b \quad \text{if and only if} \quad a \precsim_A b.$$

Basic idea of the proof Tracial strict approximate innerness allows an approximate homomorphism from $e(A \rtimes_\alpha G)e$ into eAe.

Theorem Let A be an infinite-dimensional simple unital stably finite C*-algebra and let $\alpha\colon G \to \mathrm{Aut}(A)$ be a tracially strictly approximately inner action of a finite group G on A. Then:

- $\mathrm{rc}(A) \leq \mathrm{rc}\left(A \rtimes_\alpha G\right)$.
- If $A \rtimes_\alpha G$ is simple, then $\mathrm{rc}(A) \leq \mathrm{rc}\left(A \rtimes_\alpha G\right) \leq \mathrm{rc}(A^\alpha)$.

Application

By putting the results of [2] and the above theorem together, we get the following proposition.

Proposition Let A be an infinite-dimensional simple unital stably finite C*-algebra with $0 < \mathrm{rc}(A) < \infty$. Then there is no action of any nontrivial finite group on A which both has the weak tracial Rokhlin property and is tracially strictly approximately inner.

Examples

Theorem For every finite group G and for every $\eta \in \left(0, \frac{1}{\mathrm{card}(G)}\right)$, there exist a simple separable unital AH algebra A with stable rank one and an action $\alpha\colon G \to \mathrm{Aut}(A)$ such that:

- α is pointwise outer and strictly approximately inner.
- $\mathrm{rc}(A) = \mathrm{rc}\left(A \rtimes_\alpha G\right) = \eta$.

Basic idea of the proof Our construction is given as a direct limit. Our direct system looks like the following diagram, in which the solid arrows represent many partial maps and the dotted arrows represent a small number of point evaluations:

$$C\left(X_1, M_{|G|}\right) \otimes M_{r(1)} \rightrightarrows C\left(X_2, M_{|G|}\right) \otimes M_{r(2)} \rightrightarrows \cdots$$

For $n \in \mathbb{Z}_{\geq 0}$, define $\alpha^{(n)}\colon G \to \mathrm{Aut}\left(C\left(X_n, M_{|G|}\right) \otimes M_{r(n)}\right)$ by

$$\alpha_g^{(n)}(f \otimes c) = \mathrm{Ad}\left(1_{C(X_n)} \otimes z_g\right)(f) \otimes c$$

for $f \in C(X_n, M_{|G|})$, for $c \in M_{r(n)}$, for $g \in G$, and where $z\colon G \to \mathrm{U}(l^2(G))$ is the left regular representation.

Future Plan

- One of the interesting open problems is to think of comparison theory in a simple C*-algebra A and in the crossed product of A by a discrete infinite group.

References

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M. Ali Asadi-Vasfi

- Ph.D. student, expected graduation 2020-2021
- University of Tehran

Advisors: N. Christopher Phillips and Mohammad B. Asadi

Email: Asadi.ali@ut.ac.ir

General interest: Operator algebras.