Pathwise integration over rough paths for model-free finance

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The mathematical finance formalism

In the standard formalism of mathematical finance, the price of a risky asset over a time interval $[0,T]$ is modelled by a semimartingale $Y = (Y_t)_{0\leq t\leq T}$ on a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{0\leq t\leq T}, \mathbb{P})$.

At this point, the undergraduate student may feel discouraged by the appearance of high-tech probabilistic machinery to discuss financial questions. The learned reader might ask: ``what if I picked the wrong semimartingale?"

Pedagogy and Occam's razor nudge us towards a probability-free formulation of finance.

Figure 1:Which semimartingale do we choose for TSLA? (Source: Bloomberg)

A strictly pathwise stochastic integral

It is known that stock paths of bounded variation admit arbitrage. This means we cannot define a pathwise integral as a Lebesgue-Stieltjes integral.

Instead, let us fix a refining sequence of partitions $\pi = (\pi_n)_{n=1}^{\infty}$ with vanishing mesh; given a continuous trajectory $Y \in C[0,T]$ we define its quadratic variation over π as:

$$\langle Y \rangle_t = \lim_{n\to\infty} \sum_{\substack{t_k \in \pi_n \\ t_k \leq t}} (Y_{t_{k+1}} - Y_{t_k})^2 \tag{1}$$

Whenever $t \mapsto \langle Y \rangle_t$ is continuous, we say $Y \in QV_\pi[0,T]$. We can define a pathwise Itô formula (i.e. change-of-variables) for a function $f \in C^2(\mathbb{R})$ [6]:

$$f(Y_t) - f(Y_0) = \underbrace{\int_0^t f'(Y_s) dY_s}_{\text{FTC}} + \underbrace{\frac{1}{2}\int_0^t f''(Y_s) d\langle Y \rangle_s}_{\text{QV correction}} \tag{2}$$

The FTC term can be defined as the limit of non-anticipating Riemann sums: $\int_0^t f'(Y_s)dY_s = \lim_{n\to\infty} \sum_{\pi_n} f'(Y_{t_k})(Y_{t_{k+1}} - Y_{t_k})$

Remark: $QV[0,T]$ is not a vector space and the quadratic variation $\langle Y \rangle_t$ depends on choice of π.

Applications and extensions of pathwise Itô calculus

- Black-Scholes and pathwise hedging of payoffs with local volatility [1]
- Pathwise Tanaka formula [7]
- Hedging and pricing formulas for exotics [10, 4]
- Functional pathwise Itô calculus [2, 5]
- Pathwise Itô formula for paths with arbitrary regularity [3]

Uniqueness of quadratic variation

Since the expression $\langle Y \rangle_t$ depends on the choice of partition π, it is profitable to find conditions under which QV is uniquely determined. First, we define a sequence of *balanced partitions* π:

$$\text{There exists a } c > 0 \text{ such that for all } n \geq 1 \text{ we have } \frac{\sup_{t_k\in\pi_n} |t_{k+1} - t_k|}{\inf_{t_k\in\pi_n} |t_{k+1} - t_k|} \leq c \tag{3}$$

Theorem (Cont & Das, 2017). Let π and τ be balanced partition sequences with vanishing mesh and $Y \in C^\alpha[0,T]$ for some $0 < \alpha < 1$. Suppose $Y \in QV_\pi[0,T] \cap QV_\tau[0,T]$ with $\langle Y \rangle_t$ strictly increasing under π and τ. Then, under certain regularity and roughness conditions, for all $t \in [0,T]$:

$$\langle Y \rangle_t^\pi = \langle Y \rangle_t^\tau$$

Morally, this theorem tells us that there is a space of functions on which we can find an intrinsic definition of the pathwise Itô integral.

Research question: Can this theorem be extended for arbitrary p-variation?

Do asset prices admit quadratic variation?

Gatheral, Jaisson, and Rosenbaum [8] recently observed that empirical paths of daily realised volatility time series are rougher than Brownian motion.

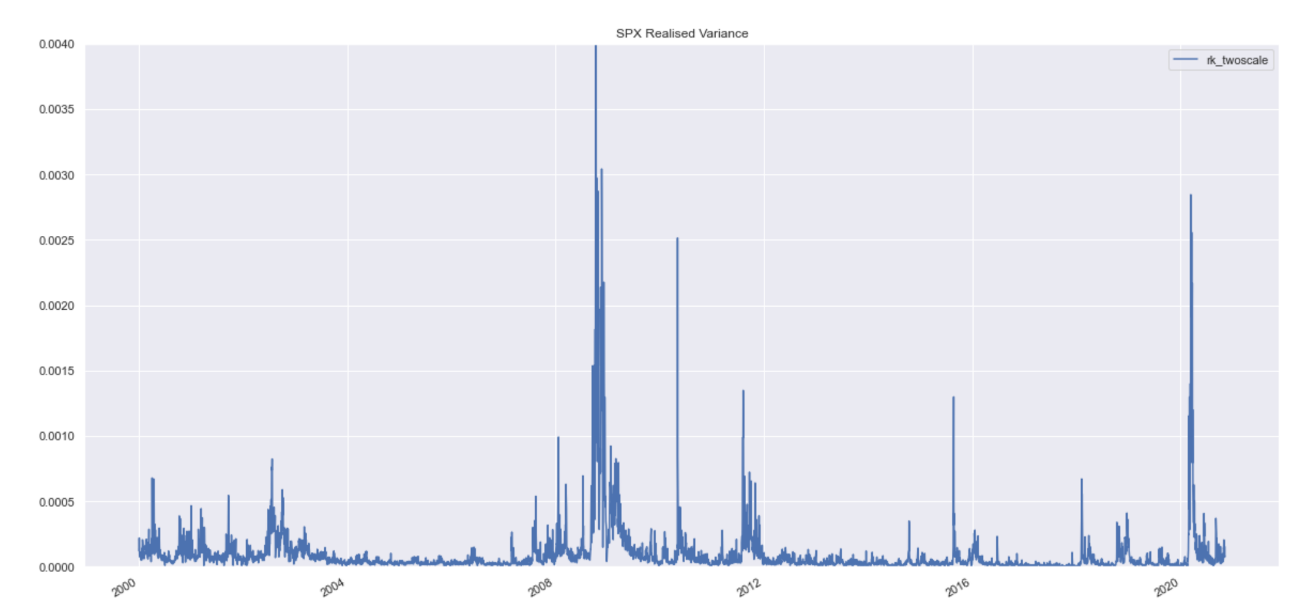

Figure 2:Time series for daily realised volatility of the SPX.

Mathematically, this means that usual paths have Hölder regularity and Hurst parameters lower than $\frac{1}{2} - \epsilon$. Their quadratic variation is infinite! Their paths are better modelled by fractional Brownian motion with Hurst index H, which admits p-variation for $p = \frac{1}{H}$.

Standard Itô calculus of semimartingales is no longer possible for these paths. A potential solution is to use the extension of Föllmer's integral by Cont and Perkowski.

Theorem. Let $p \in \mathbb{Z}^+$ be even, π a sequence of partitions, and $f \in C^{(}\mathbb{R})$. If Y admits p-variation on π (defined similarly as quadratic variation), then:

$$f(Y_t) - f(Y_0) = \int_0^t f'(Y_s) dY_s + \frac{1}{p!}\int_0^t f^{(p)}(Y_s) d\langle Y \rangle_s^{(p)} \tag{4}$$

The definition of $\int_0^t f'(Y_s)dY_s$ is a certain compensated Riemann sum:

$$\int_0^t f'(Y_s)dY_s = \lim_{n\to\infty} \sum_{t_k\in\pi_n} \sum_{r=1}^{p-1} \frac{f^{(r)}(Y_{t_k})}{r!} (Y_{t_{k+1}} - Y_{t_k})^r \tag{5}$$

Pathwise models for (fractional) Brownian paths

It is useful to study classes of continuous functions whose paths have usual features of (fractional) Brownian motion: (i) nowhere differentiable, (ii) modulus of continuity, (iii) pathwise $\frac{1}{H}$-variation.

Takagi-Landsberg functions

Define the Faber-Schauder basis $e_{0,0} = (\min(t, 1-t))^+$, $e_{m,k} = 2^{-m/2} e_{0,0}(2^m t - k)$. Set:

$$Y_t^H = \sum_{m=0}^{\infty} 2^{m(1/2 - H)} \sum_{k=0}^{2^m - 1} e_{m,k}(t) \tag{6}$$

$Y^H \in C[0,T]$ is called the Takagi-Landsberg function with Hurst index H.

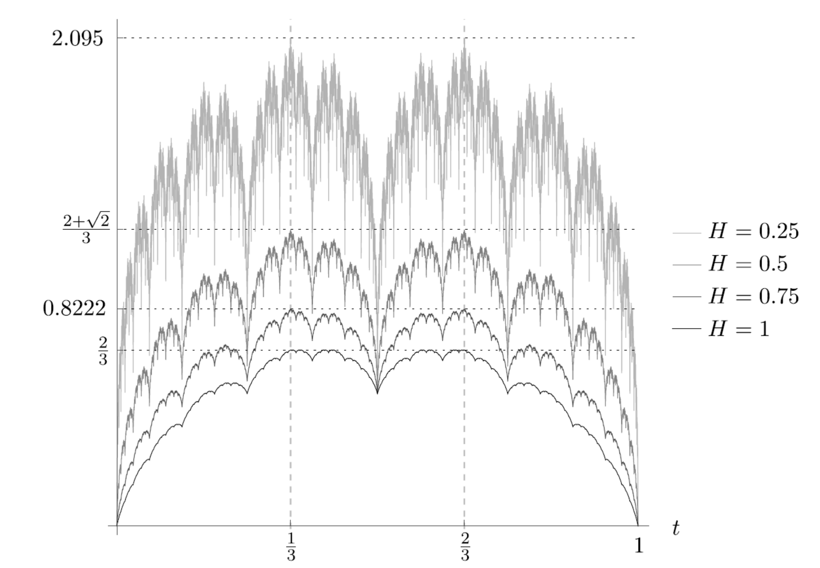

Figure 3:Time series for Takagi-Landsberg paths with different Hurst indices. (Source: [9])

The Takagi-Landsberg function has several exciting properties: it is nowhere differentiable, it has strictly increasing $\frac{1}{H}$-variation, it shares characteristics with Brownian bridges.

Research question: Which properties of the Takagi-Landsberg function make it amenable to analysis under a rough Itô pathwise calculus?

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