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Bessel pairs and optimal Hardy and Hardy–Rellich inequalities

We give necessary and sufficient conditions on a pair of positive radial functions V and W on a ball B of radius R in R^n, $n \geq 1$, so that the following inequalities hold for all $u \in C_0^\infty(B)$:

$$\int_B V(x)|\nabla u|^2\,dx \geq \int_B W(x)u^2\,dx,$$

and

$$\int_B V(x)|\Delta u|^2\,dx \geq \int_B W(x)|\nabla u|^2\,dx + (n-1)\int_B \Big(\frac{V(x)}{|x|^2} - \frac{V_r(|x|)}{|x|}\Big)|\nabla u|^2\,dx.$$

This characterization makes a very useful connection between Hardy-type inequalities and the oscillatory behaviour of certain ordinary differential equations, and helps in the identification of a large number of such couples (V, W)—that we call Bessel pairs—as well as the best constants in the corresponding inequalities. This allows us to improve, extend, and unify many results—old and new—about Hardy and Hardy–Rellich type inequalities, such as those obtained by Caffarelli–Kohn–Nirenberg, Brezis–Vázquez, Wang–Willem, Adimurthi–Chaudhuri–Ramaswamy, Filippas–Tertikas, Adimurthi–Grossi–Santra, Tertikas–Zographopoulos, and Blanchet–Bonforte–Dolbeault–Grillo–Vasquez. As an application we give a mathematical proof for the singularity of the extremal solution of the bilaplacian with exponential nonlinearity in dimensions $N \geq 13$.