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Toric Ideals Complete Intersection of Oriented Graphs and Generalized-Theta Graphs

Let G be a connected graph with n vertices and q edges and let $\mathcal{O}$ be an orientation of the edges of G, i.e., an assignment of a direction to each edge of G. Thus $\mathcal{D} = (G, \mathcal{O})$ is an *oriented graph*. To each oriented edge $e = (x_i, x_j)$ of $\mathcal{D}$, we associate the vector v_e defined as follows: the i-th entry is -1, the j-th entry is 1, and the remaining entries are zero. The *incidence matrix* $A_{\mathcal{D}}$ of $\mathcal{D}$ is the $n \times q$ matrix with entries in $\{0, \pm 1\}$ whose columns are the vectors of the form v_e, with e an edge of $\mathcal{D}$. For simplicity of notation we set $A = A_{\mathcal{D}}$. The set of column vectors of A will be denoted by $\mathcal{A} = \{v_1, \ldots, v_q\}$.

Consider the *edge subring* $k[\mathcal{D}] := k[x^{v_1}, \ldots, x^{v_q}] \subset k[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$ of the oriented graph $\mathcal{D}$. There is an epimorphism of k-algebras

$$\varphi\colon B = k[t_1, \ldots, t_q] \longrightarrow k[\mathcal{D}], \quad t_i \longmapsto x^{v_i},$$

where B is a polynomial ring. The kernel of φ, denoted by $P_{\mathcal{D}}$, is called the *toric ideal* of $\mathcal{D}$. This ideal was studied in [2], [3]. Notice that $P_{\mathcal{D}}$ is no longer a graded ideal, see Proposition **??**. The toric ideal $P_{\mathcal{D}}$ is a prime ideal of height $q-n+1$ generated by binomials and $k[\mathcal{D}]$ is a normal domain. Thus any minimal generating set of $P_{\mathcal{D}}$ must have at least $q-n+1$ elements, by the principal ideal theorem. If $P_{\mathcal{D}}$ can be generated by $q-n+1$ polynomials it is called a complete intersection. In [3] is shown that any graph has an acyclic orientation such that the corresponding toric ideal is a complete intersection. And a graph G is called complete intersection for all orientation (C.I.O.) if P_D is a complete intersection, for all D orientation of G.

We introduce the generalized-theta graph. The theta graphs studied in [1] are generalized-theta graphs. Our main result is: G is C.I.O. if and only if all generalized thetas of G have a special triangle. We obtain a characterization of the ring graphs in term of the generalized theta graph. With this result we recover the characterization of the C.I.O. bipartite graphs given in [3].

References

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[2] I. Gitler, E. Reyes and R. H. Villarreal, *Ring graphs and toric ideals.* Electron. Notes Discrete Math. **28C**(2007), 393–400.

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