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A Fractal Plancherel Theorem

A measure μ on $\mathbb{R}^n$ is called locally and uniformly h-dimensional if $\mu\big(B_r(x)\big) \leq h(r)$ for all $x \in \mathbb{R}^n$ and for all $0 < r < 1$, where h is a real valued function. If $f \in L^2(\mu)$ and $\mathcal{F}_\mu f$ denotes its Fourier transform with respect to μ, it is not true that $\mathcal{F}_\mu f \in L^2$.

We prove that, under certain hypothesis on h, for any $f \in L^2(\mu)$ the L^2-norm of its Fourier transform restricted to a ball of radius r has the same order of growth as $r^n h(r^{-1})$, when $r \to \infty$. Moreover, we prove that the ratio between these quantities is bounded by the $L^2(\mu)$-norm of f:

$$\sup_{x \in \mathbb{R}^n} \sup_{r \geq 1} \frac{1}{r^n h(r^{-1})} \int_{B_r(x)} |\mathcal{F}_\mu f(\xi)|^2 \, d\xi \leq C \|f\|_2^2.$$

By imposing certain restrictions on the measure μ, we can also obtain a lower bound for this ratio:

$$\liminf_{r \to \infty} \frac{1}{r^n h(r^{-1})} \int_{B_r(y)} |\mathcal{F}_\mu f(\xi)|^2 \, d\xi \geq c \int_E |f|^2 \, d\,\mathrm{hau}^h \, .$$

These results generalize the ones obtained by Strichartz in [1] where he considered the particular case in which $h(x) = x^\alpha$.

References

[1] R. Strichartz, *Fourier asymptotics of fractal measures.* J. Funct. Anal. **89**(1990), 154–187.