THE 1994 ASIAN PACIFIC MATHEMATICAL OLYMPIAD

Time allowed: 4 hours
NO calculators are to be used.
Each question is worth seven points.

Question 1

Let $f : \mathbb{R} \to \mathbb{R}$ be a function such that

(i) For all $x, y \in \mathbb{R}$,

$$f(x) + f(y) + 1 \geq f(x+y) \geq f(x) + f(y),$$

(ii) For all $x \in [0,1)$, $f(0) \geq f(x)$,

(iii) $-f(-1) = f(1) = 1$.

Find all such functions f.

Question 2

Given a nondegenerate triangle ABC, with circumcentre O, orthocentre H, and circumradius R, prove that $|OH| < 3R$.

Question 3

Let n be an integer of the form $a^2 + b^2$, where a and b are relatively prime integers and such that if p is a prime, $p \leq \sqrt{n}$, then p divides ab. Determine all such n.

Question 4

Is there an infinite set of points in the plane such that no three points are collinear, and the distance between any two points is rational?

Question 5

You are given three lists A, B, and C. List A contains the numbers of the form 10^k in base 10, with k any integer greater than or equal to 1. Lists B and C contain the same numbers translated into base 2 and 5 respectively:

A	B	C
10	1010	20
100	1100100	400
1000	1111101000	13000
$\vdots$	$\vdots$	$\vdots$

Prove that for every integer $n > 1$, there is exactly one number in exactly one of the lists B or C that has exactly n digits.