THE 1990 ASIAN PACIFIC MATHEMATICAL OLYMPIAD

Time allowed: 4 hours
NO calculators are to be used.
Each question is worth seven points.

Question 1

Given triagnle ABC, let D, E, F be the midpoints of BC, AC, AB respectively and let G be the centroid of the triangle.
For each value of $\angle BAC$, how many non-similar triangles are there in which $AEGF$ is a cyclic quadrilateral?

Question 2

Let a_1, a_2, $\ldots$, a_n be positive real numbers, and let S_k be the sum of the products of a_1, a_2, $\ldots$, a_n taken k at a time. Show that

$$S_k S_{n-k} \geq \binom{n}{k}^2 a_1 a_2 \cdots a_n$$

for $k = 1, 2, \ldots, n-1$.

Question 3

Consider all the triangles ABC which have a fixed base AB and whose altitude from C is a constant h. For which of these triangles is the product of its altitudes a maximum?

Question 4

A set of 1990 persons is divided into non-intersecting subsets in such a way that

1. No one in a subset knows all the others in the subset,

2. Among any three persons in a subset, there are always at least two who do not know each other, and

3. For any two persons in a subset who do not know each other, there is exactly one person in the same subset knowing both of them.

(a) Prove that within each subset, every person has the same number of acquaintances.

(b) Determine the maximum possible number of subsets.
Note: It is understood that if a person A knows person B, then person B will know person A; an acquaintance is someone who is known. Every person is assumed to know one's self.

Question 5

Show that for every integer $n \geq 6$, there exists a convex hexagon which can be dissected into exactly n congruent triangles.