Solutions For March

668. The nonisosceles right triangle ABC has $\angle CAB = 90^\circ$. The inscribed circle with centre T touches the sides AB and AC at U and V respectively. The tangent through A of the circumscribed circle meets UV produced in S. Prove that

(a) $ST \parallel BC$;

(b) $|d_1 - d_2| = r$, where r is the radius of the inscribed circle and d_1 and d_2 are the respective distances from S to AC and AB.

(a) *Solution 1.* Wolog, suppose that the situation is as diagrammed. $\angle BAC = \angle AUT = \angle AVT = 90^\circ$, so that $AUVT$ is a rectangle with $AU = AV$ and $UT = VT$. Hence $AUTV$ is a square with diagonals AT and UV which right-bisect each other at W. Since SW right-bisects AT, by reflection in the line SW, we see that $\Delta ASU \equiv \Delta UST$, and so $\angle UTS = \angle UAS$.

Let M be the midpoint of BC. Then M is the circumcentre of ΔABC, so that $MA = MC$ and $\angle MCA = \angle MAC$. Since AS is tangent to the circumcircle of ΔABC, $AS \perp AM$. Hence

$$\angle UTS = \angle UAS = \angle SAM - \angle BAM = 90^\circ - \angle BAM = \angle MAC = \angle MCA \ .$$

Now $UT \perp AB$ implies that $UT \| AC$. Since $\angle UTS = \angle ACB$, it follows that $ST \| BC$.

Solution 2. Wolog, suppose that S is on the opposite side of AB to C.

BT, being a part of the diameter produced of the inscribed circle, is a line of reflection that takes the circle to itself and takes the tangent BA to BC. Hence $\angle UBT = \frac{1}{2}\angle ABC$. Let $\alpha = \angle ABT$. By the tangent-chord theorem applied to the circumscribed circle, $\angle XAC = \angle ABC = 2\alpha$, so that $\angle SAU = 90^\circ - 2\alpha$.

Consider triangles SAU and STU. Since $AUTV$ is a square (see the first solution), $AU = UT$ and $\angle AUV = \angle TUV = 45^\circ$ so $\angle SUA = \angle SUT = 135^\circ$. Also SU is common. Hence $\Delta SAU \equiv \Delta STU$, so $\angle STU = \angle SAU = 90^\circ - 2\alpha$. Therefore,

$$\angle STB = \angle UTB - \angle STU = (90^\circ - \alpha) - (90^\circ - 2\alpha) = \alpha = \angle TBC$$

from which it results that $ST \| BC$.

Solution 3. As before $\Delta AUS \equiv \Delta TUS$, so $\angle SAU = \angle STU$. Since $UT \| AC$, $\angle STU = \angle SYA$. Also, by the tangent-chord theorem, $\angle SAB = \angle ACB$. Hence $\angle SYA = \angle STU = \angle SAB = \angle ACB$, so $ST \| BC$.

Solution 4. In the Cartesian plane, let $A \sim (0, 0)$, $B \sim (0, -b)$, $C \sim (c, 0)$. The centre of the circumscribed circle is at $M \sim (c/2, -b/2)$. Since the slope of AM is $-b/c$, the equation of the tangent to the circumscribed circle through A is $y = (c/b)x$. Let r be the radius of the inscribed circle. Since $AU = AV$, the equation of the line UV is $y = x - r$. The abscissa of S is the solution of $x - r = (cx)/b$, so $S \sim (\frac{br}{b-c}, \frac{cr}{b-c})$. Since $T \sim (r, -r)$, the slope of ST is b/c and the result follows.

(b) *Solution 1.* $[\cdots]$ denotes area. Wolog, suppose that $d_1 > d_2$, as diagrammed.

Let r be the inradius of ΔABC. Then $[AVU] = \frac{1}{2}r^2$, $[AVS] = \frac{1}{2}rd_1$ and $[AUS] = \frac{1}{2}rd_2$. From $[AVU] = [AVS] - [AUS]$, it follows that $r^2 = rd_1 - rd_2$, whence $r = d_1 - d_2$.

Solution 2. [F. Crnogorac] Suppose that the situation is as diagrammed. Let P and Q be the respective feet of the perpendiculars from S to AC and AB. Since $\angle PVS = 45^\circ$ and $\angle SPV = 90^\circ$, ΔPSV is isosceles and so $PS = PV = PA + AV = SQ + AV$, *i.e.*, $d_1 = d_2 + r$.

Solution 3. Using the coordinates of the fourth solution of (a), we find that

$$d_1 = \left| \frac{cr}{b-c} \right| \qquad \text{and} \qquad d_2 = \left| \frac{br}{b-c} \right|$$

whence $|d_2 - d_1| = r$ as desired.

(b) *Solution.* [M. Boase] Wolog, assume that the configuration is as diagrammed.

Since $\angle SUB = \angle AUV = 45°$, SU is parallel to the external bisector of $\angle A$. This bisector is the locus of points equidistant from AB and CA produced. Wolog, let PS meet this bisector in W, as in the diagram. Then $PW = PA$ so that $PS - PA = PS - PW = SW = AU$ and thus $d_1 - d_2 = r$.

669. Let $n \geq 3$ be a natural number. Prove that

$$1989 | n^{n^{n^n}} - n^{n^n} \quad ,$$

i.e., the number on the right is a multiple of 1989.

Solution 1. Let $N = n^{n^{n^n}} - n^{n^n}$. Since $1989 = 3^2 \cdot 13 \cdot 17$,

$$N \equiv 0 \pmod{1989} \Leftrightarrow N \equiv 0 \pmod{9, 13 \ \& \ 17} \ .$$

We require the following facts:
(i) $x^u \equiv 0 \pmod 9$ whenever $u \geq 2$ and $x \equiv 0 \pmod 3$.
(ii) $x^6 \equiv 1 \pmod 9$ whenever $x \not\equiv 0 \pmod 3$.
(iii) $x^u \equiv 0 \pmod{13}$ whenever $x \equiv 0 \pmod{13}$.
(iv) $x^{12} \equiv 1 \pmod{13}$ whenever $x \not\equiv 0 \pmod{13}$, by Fermat's Little Theorem.
(v) $x^u \equiv 0 \pmod{17}$ whenever $x \equiv 0 \pmod{17}$.
(vi) $x^{16} \equiv 1 \pmod{17}$ whenever $x \not\equiv 0 \pmod{17}$, by FLT.
(vii) $x^4 \equiv 1 \pmod{16}$ whenever $x = 2y + 1$ is odd. (For, $(2y+1)^4 = 16y^3(y+2) + 8y(3y+1) + 1 \equiv 1 \pmod{16}$.)

Note that

$$N = n^{n^n}\left[n^{(n^{n^n} - n^n)} - 1\right] = n^{n^n}\left[n^{n^n(n^{n^n - n} - 1)} - 1\right] .$$

Modulo 17. If $n \equiv 0 \pmod{17}$, then $n^{n^n} \equiv 0$, and so $N \equiv 0 \pmod{17}$.

If n is even, $n \geq 4$, then $n^n \equiv 0 \pmod{16}$, so that

$$n^{n^n(n^{n^n - n} - 1)} \equiv 1^{(n^{n^n - n} - 1)} \equiv 1$$

so $N \equiv 0 \pmod{17}$.

Suppose that n is odd. Then $n^n \equiv n \pmod 4$

$$\Rightarrow n^n - n = 4r \ \text{ for some } \ r \in \mathbf{N}$$

$$\Rightarrow n^{n^n - n} = n^{4r} \equiv 1 \pmod{16}$$

$$\Rightarrow n^{n^n - n} - 1 \equiv 0 \pmod{16}$$

$$\Rightarrow n^{n^n(n^{n^n - n} - 1)} \equiv 1 \ \ (\text{mod } 17$$

$$\Rightarrow N \equiv 0 \pmod{17} \ .$$

Hence $N \equiv 0 \pmod{17}$ for all $n \geq 3$.

Modulo 13. If $n \equiv 0 \pmod{13}$, then $n^{n^n} \equiv 0$ and $N \equiv 0 \pmod{13}$.

Suppose that n is even. Then $n^n \equiv 0 \pmod 4$, so that $n^{n^n} - n^n \equiv 0 \pmod 4$. Suppose that n is odd. Then $n^{n^n - n} - 1 \equiv 0 \pmod{16}$ and so $n^{n^n} - n^n \equiv 0 \pmod 4$.

If $n \equiv 0 \pmod 3$, then $n^n \equiv 0$ so $n^n(n^{n^n-n} - 1) \equiv 0 \pmod 3$. If $n \equiv 1 \pmod 3$, then $n^{n^n-n} \equiv 1$ so $n^n(n^{n^n-n}-1) \equiv 0 \pmod 3$. If $n \equiv 2 \pmod 3$, then, as $n^n - n$ is always even, $n^{n^n-n} \equiv 1$ so $n^n(n^{n^n-n}-1) \equiv 0 \pmod 3$. Hence, for all n, $n^{n^n} - n^n \equiv 0 \pmod 3$.

It follows that $n^{n^n} - n^n \equiv 0 \pmod{12}$ for all values of n. Hence, when n is not a multiple of 13, $n^{(n^{n^n}-n)} \equiv 1$ so $N \equiv 0 \pmod{13}$.

Modulo 9. If $n \equiv 0 \pmod 3$, then $n^{n^n} \equiv 0 \pmod 9$, so $N \equiv 0 \pmod 9$. Let $n \not\equiv 0 \pmod 9$. Since $n^{n^n} - n^n$ is divisible by 12, it is divisible by 6, and so $n^{(n^{n^n}-n^n)} \equiv 1$ and $N \equiv 0 \pmod 9$. Hence $N \equiv 0 \pmod 9$ for all n.

The required result follows.

670. Consider the sequence of positive integers $\{1, 12, 123, 1234, 12345, \cdots\}$ where the next term is constructed by lengthening the previous term at the right-hand end by appending the next positive integer. Note that this next integer occupies only one place, with "carrying"occurring as in addition. Thus, the ninth and tenth terms of the sequence are 123456789 and 1234567900 respectively. Determine which terms of the sequence are divisible by 7.

Solution 1. For positive integer n, let x_n be the nth term of the sequence, and let $x_0 = 0$. Then, for $n \geq 0$, $x_{n+1} = 10x_n + (n+1)$ so that $x_{n+1} \equiv 3x_n + (n+1) \pmod 7$. Suppose that m is a nonnegative integer and that $x_{7m} = a$. Then

$$x_{7m+1} \equiv 3a+1 \qquad x_{7m+2} \equiv 2a+5 \qquad x_{7m+3} \equiv 6a+4 \qquad x_{7m+4} \equiv 4a+2$$
$$x_{7m+5} \equiv 5a+4 \qquad x_{7m+6} \equiv a+4 \qquad x_{7m+7} \equiv 3a+5$$

In particular, we find that, modulo 7, $\{x_{7m}\}$ is periodic with the values $\{0, 5, 6, 2, 4, 3\}$ repeated, so that $0 \equiv x_0 \equiv x_{42} \equiv x_{84} \equiv \cdots$. Hence, modulo 7, $x_{7m+1} \equiv 0$ iff $a \equiv 2$, $x_{7m+2} \equiv 0$ iff $a \equiv 1$, $x_{7m+3} \equiv 0$ iff $a \equiv 4$, $x_{7m+4} \equiv 0$ iff $a \equiv 3$, $x_{7m+5} \equiv 0$ iff $a \equiv 2$ and $x_{7m+6} \equiv 0$ iff $a \equiv 3$. Putting this all together, we find that $x_n \equiv 0 \pmod 7$ if and only if $n \equiv 0, 22, 26, 31, 39, 41 \pmod{42}$.

Solution 2. [C. Deng] Recall the formula

$$r^{n-1} + 2r^{n-2} + \cdots + (n-1)r + n = \frac{r^{n+1} - r - (r-1)n}{(r-1)^2} \ .$$

[Derive this.] Noting that

$$a_n = 1 \cdot 10^{n-1} + 2 \cdots 10^{n-2} + \cdots + (n-1) \cdot 10 + n \ ,$$

we find that

$$81a_n = 10^{n+1} - 10 - 9n$$

for each positive integer n. Therefore

$$81(a_{n+42} - a_n) = 10^{n+1}((10^6)^7 - 1) - 9(42)$$

for each positive integer n. Since $10^6 \equiv 1$ (modulo 7), it follows that $a_{n+42} \equiv a_n$ (modulo 7), so that the sequence has period 42 (modulo 7). Thus, the value of n for which a_n is divisible by 7 are the solutions of the congruence $3^{n+1} \equiv 2n+3$ (modulo 7). These are $n \equiv 22, 26, 31, 39, 41, 42$ (modulo 7).

671. Each point in the plane is coloured with one of three distinct colours. Prove that there are two points that are unit distant apart with the same colour.

Solution 1. Suppose that the points in the plane are coloured with three colours. Select any point P.

We form two rhombi $PQSR$ and $PUWV$, one the rotated image of the other for which all of the following segments have unit length: PQ, PR, SQ, SR, QR, PU, PV, WU, WV, UV, SW. If P, Q, R are all coloured differently, then either the result holds or S must have the same colour as P. If P, U, V are all coloured differently, then either the result holds or W must have the same colour as P. Hence, either one of the triangles PQR and PUV has two vertices the same colour, or else S and W must be coloured the same.

Solution 2. Suppose, if possible, the planar points can be coloured without two points unit distance apart being coloured the same. Then if A and B are distant $\sqrt{3}$ apart, then there are distinct points C and D such that ACD and BCD are equilateral triangles ($ACBD$ is a rhombus). Since A and B must be coloured differently from the two colours of C and D, A and B must have the same colour. Hence, if O is any point in the plane, every point on the circle of radius $\sqrt{3}$ consists of points coloured the same as O. But there are two points on this circle unit distant apart, and we get a contradiction of our initial assumption.

Solution 3. Suppose we can colour the points of the plane with three colours, red, blue and yellow so that the result fails. We show that three collinear points at unit distance are coloured with three different colours. Let P, Q, R be three such points, and let P, R be opposite sides of a unit hexagon $ABPCDR$ whose centre is Q.

If, say, Q is red, B and A must be coloured differently, as are A and R, R and D, D and C, C and P, P and B. Thus, B, R, C, are one colour, say, blue, and A, D, P the other, say yellow. The preliminary result follows.

Now consider any isosceles triangle UVW with $|UV| = |UW| = 3$ and $|VW| = 2$. It follows from the preliminary result that U and V must have the same colour, as do U and W. But V and W cannot have the same colour and we reach a contradiction.

Solution 4. [D. Arthur] Suppose that the result is false. Let A, B be two points with $|AB| = 3$. Within the segment AB select P, Q with $|AP| = |PQ| = |QB| = 1$, and suppose that R and S are points on the same side of AB with ΔRAP and ΔSPQ equilateral. Then $|RS| = 1$. Suppose if possible that A and Q have the same colour. Then P must have a second colour and R and S the third, leading to a contradiction. Hence A must be coloured differently from both P and Q. Similarly B must be coloured differently from both P and Q. Since P and Q are coloured differently, A and B must have the same colour.

Now consider a trapezoid $ABCD$ with $|CB| = |AB| = |AD| = 3$ and $|CD| = 1$. By the foregoing observation, C, A, B, D must have the same colour. But this yields a contradiction. The result follows.

672. The Fibonacci sequence $\{F_n\}$ is defined by $F_1 = F_2 = 1$ and $F_{n+2} = F_{n+1} + F_n$ for $n = 0, \pm 1, \pm 2, \pm 3, \cdots$. The real number τ is the positive solution of the quadratic equation $x^2 = x + 1$.

(a) Prove that, for each positive integer n, $F_{-n} = (-1)^{n+1} F_n$.

(b) Prove that, for each integer n, $\tau^n = F_n \tau + F_{n-1}$.

(c) Let G_n be any one of the functions $F_{n+1}F_n$, $F_{n+1}F_{n-1}$ and F_n^2. In each case, prove that $G_{n+3} + G_n = 2(G_{n+2} + G_{n+1})$.

(a) *Solution.* Since $F_0 = F_2 - F_1 = 0$, the result holds for $n = 0$. Since $F_{-1} = F_1 - F_0 = 1$, the result holds for $n = 1$. Suppose that we have established the result for $n = 0, 1, 2, \cdots r$. Then

$$F_{-(r+1)} = F_{-r-1} = F_{-r+1} - F_{-r} = (-1)^r F_{r-1} - (-1)^{r+1} F_r = (-1)^{r+2}(F_{r-1} + F_r) = (-1)^{r+2} F_{r+1} \; .$$

The result follows by induction.

(b) *Solution 1.* The result holds for $n = 0$, $n = 1$ and $n = 2$. Suppose that it holds for $n = 0, 1, 2, \cdots, r$. Then

$$\tau^{r+1} = \tau^r + \tau^{r-1} = (F_r + F_{r-1})\tau + (F_{r-1} + F_{r-2}) = F_{r+1}\tau + F_r\tau \; .$$

This establishes the result for positive values of n. Now $\tau^{-1} = \tau - 1 = F_{-1}\tau + F_{-2}$, so the result holds for $n = -1$. Suppose that we have established the result for $n = 0, -1, -2, \cdots, -r$. Then

$$\tau^{-(r+1)} = \tau^{-(r-1)} - \tau^{-r} = (F_{-(r-1)} - F_{-r})\tau + (F_{-r} - F_{-(r+1)}) = F_{-(r+1)}\tau + F_{-(r+2)} \; .$$

Solution 2. The result holds for $n = 1$. Suppose that it holds for $n = r \geq 0$. Then

$$\tau^{r+1} = \tau^r \cdot \tau = (F_r \tau + F_{r-1})\tau = F_r \tau^2 + F_{r-1}\tau$$

$$= (F_r + F_{r-1})\tau + F_r = F_{r+1}\tau + F_r \quad .$$

Now consider nonpositive values of n. We have that $\tau^0 = 1$, $\tau^{-1} = \tau - 1$, $\tau^{-2} = 1 - \tau^{-1} = 2 - \tau$. Suppose that we have shown for $r \geq 0$ that $\tau^{-r} = F_{-r}\tau + F_{-r-1}$. Then

$$\tau^{-(r+1)} = \tau^{-1}\tau^{-r} = F_{-r} + F_{-r-1}(\tau - 1) = F_{-r-1}\tau + (F_{-r} - F_{-r-1})$$

$$= F_{-r-1}\tau + F_{-r-2} = F_{-(r+1)}\tau + F_{-(r+1)-1} \quad .$$

By induction, it follows that the result holds for both positive and negative values of n.

(c) *Solution.* Let $G_n = F_n F_{n+1}$. Then

$$\begin{aligned} G_{n+3} + G_n &= F_{n+4}F_{n+3} + F_{n+1}F_n \\ &= (F_{n+3} + F_{n+2})(F_{n+2} + F_{n+1}) + (F_{n+3} - F_{n+2})(F_{n+2} - F_{n+1}) \\ &= 2(F_{n+3}F_{n+2} + F_{n+2}F_{n+1}) = 2(G_{n+2} + G_{n+1}) \ . \end{aligned}$$

Let $G_n = F_{n+1}F_{n-1}$. Then

$$\begin{aligned} G_{n+3} + G_n &= F_{n+4}F_{n+2} + F_{n+1}F_{n-1} \\ &= (F_{n+3} + F_{n+2})(F_{n+1} + F_n) + (F_{n+3} - F_{n+2})(F_{n+1} - F_n) \\ &= 2(F_{n+3}F_{n+1} + F_{n+2}F_n) = 2(G_{n+2} + G_{n+1}) \ . \end{aligned}$$

Let $G_n = F_n^2$. Then

$$\begin{aligned} G_{n+3} + G_n &= F_{n+3}^2 + F_n^2 = (F_{n+2} + F_{n+1})^2 + (F_{n+2} - F_{n+1})^2 \\ &= F_{n+2}^2 + 2F_{n+2}F_{n+1} + F_{n+1}^2 + F_{n+2}^2 - 2F_{n+2}F_{n+1} + F_{n+1}^2 = 2(G_{n+2} + G_{n+1}) \ . \end{aligned}$$

Comments. Since $F_n^2 = F_nF_{n-1} + F_nF_{n-2}$, the third result of (c) can be obtained from the first two. J. Chui observed that, more generally, we can take $G_n = F_{n+u}F_{n+v}$ where u and v are integers. Then

$$\begin{aligned} &G_{n+3} + G_n - 2(G_{n+1} + G_{n+2}) \\ &\quad = (F_{n+3+u}F_{n+3+v} + F_{n+u}F_{n+v}) - 2(F_{n+2+u}F_{n+2+v} + F_{n+1+u}F_{n+1+v}) \\ &\quad = (2F_{n+1+u} + F_{n+u})(2F_{n+1+v} + F_{n+v}) + F_{n+u}F_{n+v} \\ &\qquad\quad - 2(F_{n+1+u} + F_{n+u})(F_{n+1+v} + F_{n+v}) - 2F_{n+1+u}F_{n+1+v} \\ &\quad = 0 \quad , \end{aligned}$$

so that $G_{n+3} + G_n = 2(G_{n+2} + G_{n+1})$.

673. ABC is an isosceles triangle with $AB = AC$. Let D be the point on the side AC for which $CD = 2AD$. Let P be the point on the segment BD such that $\angle APC = 90°$. Prove that $\angle ABP = \angle PCB$.

Solution 1. Produce BA to E so that $BA = AE$ and join EC. Then D is the centroid of ΔBEC and BD produced meets EC at its midpoint F. Since $AE = AC$, ΔCAE is isosceles and so $AF \perp EC$. Also, since A and F are midpoints of their respective segments, $AF \| BC$ and so $\angle AFB = \angle DBC$. Because $\angle AFC$ and $\angle APC$ are both right, $APCF$ is concyclic so that $\angle AFP = \angle ACP$.

Hence $\angle ABP = \angle ABC - \angle DBC = \angle ABC - \angle AFB = \angle ACB - \angle ACP = \angle PCB$.

Solution 2. Let E be the midpoint of BC and let F be a point on BD produced so that $AF \| BC$. Since triangle ADF and CDB are similar and $CD = 2AD$, then $AF = EC$ and $AECF$ is a rectangle.

Since $\angle APC = \angle AFC = 90°$, the quadrilateral $APCF$ is concyclic, so that $\angle AFB = \angle ACP$. Since $AF \| BC$, $\angle AFB = \angle FBC$. Therefore

$$\angle ABP = \angle ABC - \angle PBC = \angle ABC - \angle FBC = \angle ACB - \angle ACP = \angle PCB \ .$$

Solution 3. [S. Sun] The circle with diameter AC has as its centre the midpoint O of AC. It intersects BC at the midpoint E (since $AB = AC$ and $AE \perp BC$). Let EO produced meet the circle again at F; then $AECF$ is concyclic.

Suppose FB meets AC at G. A rotation of $180°$ about O takes $A \leftrightarrow C$, $F \leftrightarrow E$, so that $BC = 2EC = 2AF$ and $AF \| BC$. The triangles AGF and CGB are similar. Since $BC = 2AF$, then $CG = 2GA$, so that G and D coincide. Because $AF \| BC$ and $AFCP$ is concyclic, $\angle DBC = \angle DFA = \angle PFA = \angle PCA$. Therefore

$$\angle ABP = \angle ABC - \angle DBC = \angle ABC - \angle PCA = \angle PCB \ .$$

Solution 4. Assign coordinates: $A \sim (0, a)$, $B \sim (-1, 0)$, $C \sim (1, 0)$. Then $D \sim (\frac{1}{3}, \frac{2a}{3})$. Let $P \sim (p, q)$. Then, since P lies on the lines $y = \frac{a}{2}(x + 1)$, $q = \frac{a}{2}(p + 1)$. The relation $AP \perp PC$ implies that

$$-1 = \left(\frac{q - a}{p}\right)\left(\frac{q}{p - 1}\right) = \left[\frac{a(p - 1)}{2p}\right]\left[\frac{a(p + 1)}{2(p - 1)}\right] = \frac{a^2(p + 1)}{4p} = \frac{aq}{2p}$$

whence $p = -a^2/(a^2 + 4)$ and $q = 2a/(a^2 + 4)$. Now

$$\tan \angle ABP = \frac{a - (a/2)}{1 + (a^2/2)} = \frac{a}{2 + a^2}$$

while

$$\tan \angle PCB = \frac{-q}{p - 1} = \frac{-2a}{-a^2 - (a^2 + 4)} = \frac{a}{a^2 + 2} = \tan \angle ABP \quad .$$

The result follows.

Solution 5. [C. Deng] Let $A \sim (0, b)$, $B \sim (-a, 0)$, $C \sim (a, 0)$ so that $D \sim (a/3, 2b/3)$. The midpoint M of AC has coordinates $(a/2, b/2)$. It can be checked that the point with coordinates

$$\left(\frac{-ab^2}{4a^2 + b^2}, \frac{2a^2 b}{4a^2 + b^2}\right)$$

is the same distance from M as the points AB so that it is on the circle with diameter AC and $AP \| CP$. Since this point also lies on the line with equation $2ay = bx + ba$ through B and D, it is none other than the point P. The circle with equation

$$x^2 + \left(y + \frac{a^2}{b}\right)^2 = a^2 + \frac{a^4}{b^2}$$

is tangent to AB and AC at B and C respectively and contains the point P. Hence $\angle PCB = \angle PBA = \angle DBA$, as desired.

674. The sides BC, CA, AB of triangle ABC are produced to the poins R, P, Q respectively, so that $CR = AP = BQ$. Prove that triangle PQR is equilateral if and only if triangle ABC is equilateral.

Solution . Suppose that triangle ABC is equilateral. A rotation of $60°$ about the centroid of ΔABC will rotate the points R, P and Q. Hence ΔPQR is equilateral. On the other hand, suppose, wolog, that $a \geq b \geq c$, with $a > c$. Then, for the internal angles of ΔABC, $A \geq B \geq C$. Suppose that $|PQ| = r$, $|QR| = p$ and $|PR| = q$, while s is the common length of the extensions. Then

$$p^2 = s^2 + (a + s)^2 + 2s(a + s)\cos B$$

and

$$r^2 = s^2 + (c + s)^2 + 2s(c + s)\cos A \ .$$

Since $a > c$ and $\cos B \geq \cos A$, we find that $p > r$, and so ΔPQR is not equilateral.