PROBLEMS FOR JANUARY, 2010

Please send your solutions to

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individually as you solve the problems. Electronic files can be sent to *barbeau@math.utoronto.ca*. However, please do not send scanned files; they use a lot of computer space, are often indistinct and can be difficult to download.

It is important that your complete mailing address and your email address appear legibly on the front page. If you do not write your family name last, please underline it.

Solutions to problems 656, 657 and 658 can also be sent to the proposer Rosu Mihai, 54 Judith Crescent, Brampton, ON L6S 3J4 (*rosumihai@yahoo.ca*).

654. Let ABC be an arbitrary triangle with the points D, E, F on the sides BC, CA, AB respectively, so that

$$\frac{BD}{DC} \leq \frac{BF}{FA} \leq 1$$

and

$$\frac{AE}{EC} \leq \frac{AF}{FB} \quad .$$

Prove that $[DEF] \leq \frac{1}{4}[ABC]$, with equality if and only if two at least of the three points D, E, F are midpoints of the corresponding sides.
(*Note:* $[XYZ]$ denotes the area of triangle XYZ.)

655. (a) Three ants crawl along the sides of a fixed triangle in such a way that the centroid (intersection of the medians) of the triangle they form at any moment remains constant. Show that this centroid coincides with the centroid of the fixed triangle if one of the ants travels along the entire perimeter of the triangle.

(b) Is it indeed always possible for a given fixed triangle with one ant at any point on the perimeter of the triangle to place the remaining two ants somewhere on the perimeter so that the centroid of their triangle coincides with the centroid of the fixed triangle?

656. Let ABC be a triangle and k be a real constant. Determine the locus of a point M in the plane of the triangle for which

$$|MA|^2 \sin 2A + |MB|^2 \sin 2B + |MC|^2 \sin 2C = k \ .$$

657. Let a, b, c be positive real numbers for which $a + b + c = abc$. Find the minimum value of

$$\sqrt{1 + \frac{1}{a^2}} + \sqrt{1 + \frac{1}{b^2}} + \sqrt{1 + \frac{1}{c^2}} \ .$$

658. Prove that $\tan 20^\circ + 4 \sin 20^\circ = \sqrt{3}$.

659. (a) Give an example of a pair a, b of positive integers, not both prime, for which $2a - 1$, $2b - 1$ and $a + b$ are all primes. Determine all possibilities for which a and b are themselves prime.

(b) Suppose a and b are positive integers such that $2a - 1$, $2b - 1$ and $a + b$ are all primes. Prove that neither $a^b + b^a$ nor $a^a + b^b$ are multiples of $a + b$.

660. ABC is a triangle and D is a point on AB produced beyond B such that $BD = AC$, and E is a point on AC produced beyond C such that $CE = AB$. The right bisector of BC meets DE at P. Prove that $\angle BPC = \angle BAC$.