Solutions for November

647. Find all continuous functions $f : \mathbf{R} \rightarrow \mathbf{R}$ such that

$$f(x + f(y)) = f(x) + y$$

for every $x, y \in \mathbf{R}$.

Solution 1. Setting $(x, y) = (t, 0)$ yields $f(t + f(0)) = f(t)$ for all real t. Setting $(x, y) = (0, t)$ yields $f(f(t)) = f(0) + t$ for all real t. Hence $f(f(f(t))) = f(t)$ for all real t, *i.e.*, $f(f(z)) = z$ for each z in the image of f. Let $(x, y) = (f(t), -f(0))$. Then

$$f(f(t) + f(-f(0))) = f(f(t)) - f(0) = f(0) + t - f(0) = t$$

so that the image of f contains every real and so $f(f(t)) \equiv t$ for all real t.

Taking $(x, y) = (u, f(v))$ yields

$$f(u + v) = f(u) + f(v)$$

since $v = f(f(v))$ for all real u and v. In particular, $f(0) = 2f(0)$, so $f(0) = 0$ and $0 = f(-t + t) = f(-t) + f(t)$. By induction, it can be shown that for each integer n and each real t, $f(nt) = nf(t)$. In particular, for each rational r/s, $f(r/s) = rf(1/s) = (r/s)f(1)$. Since f is continuous, $f(t) = f(t \cdot 1) = tf(1)$ for all real t. Let $c = f(1)$. Then $1 = f(f(1)) = f(c) = cf(1) = c2$ so that $c = \pm 1$. Hence $f(t) \equiv t$ or $f(t) \equiv -t$. Checking reveals that both these solutions work. (For $f(t) \equiv -t$, $f(x + f(y)) = -x - f(y) = f(x) + y$, as required.)

Solution 2. Taking $(x, y) = (0, 0)$ yields $f(f(0)) = f(0)$, whence $f(f(f(0))) = f(f(0)) = f(0)$. Taking $(x, y) = (0, f(0))$ yields $f(f(f(0))) = 2f(0)$. Hence $2f(0) = f(0)$ so that $f(0) = 0$. Taking $x = 0$ yields $f(f(y)) = y$ for each y. We can complete the solution as in the Second Solution.

Solution 3. [J. Rickards] Let $(x, y) = (x, -f(x))$ to get

$$f(x + f(-f(x)) = f(x) - f(x) = 0$$

for all x. Thus, there is at least one element u for which $f(u) = 0$. But then, taking $(x, y) = (0, u)$, we find that $f(0) = f(0 + f(u)) = f(0) + u$, so that $u = 0$.

Therefore $f(f(y)) = y$ for each y, so that f is a one-one onto function. Also, $x + f(-f(x)) = 0$, so that $-f(x) = f(f(-f(x)) = f(-x)$ for each value of x.

Since $f(x)$ is continuous and vanishes only for $x = 0$, we have either (1) $f(x)$ is positive for $x > 0$ and negative for $x < 0$, or (2) $f(x)$ is negative for $x > 0$ and positive for $x < 0$. Suppose that situation (1) obtains. Then, for every real number x, $f(x - f(x)) = f(x + f(-x)) = f(x) - x = -(x - f(x))$. Since $f(x - f(x))$ and $x - f(x)$ have the same sign, we must have $f(x) = x$. Suppose that situation (2) obtains. Then, for every real x, $f(x + f(x)) = f(x) + x$, from which we deduce that $f(x) = -x$. Therefore, there are two functions $f(x) = x$ and $f(x) = -x$ that satisfy the equation and both work.

648. Prove that for every positive integer n, the integer $1 + 5^n + 5^{2n} + 5^{3n} + 5^{4n}$ is composite.

Solution. Observe the following representations:

$$x8 + x6 + x4 + x2 + 1 = (x4 + x3 + x2 + x + 1)(x4 - x3 + x2 - x + 1)\ . \qquad (1)$$

and

$$x4 + x3 + x2 + x + 1 = (x2 + 3x + 1)2 - 5x(x + 1)2\ . \qquad (2)$$

When $n = 2k$ is even, we can substitute $x = 5^k$ into equation (1) to get a factorization. When $n = 2k - 1$ is odd, we can substitute $x = 5^{2k-1}$ into equation (2) to get a difference of squares, which can then be factored.

649. In the triangle ABC, $\angle BAC = 20^\circ$ and $\angle ACB = 30^\circ$. The point M is located in the interior of triangle ABC so that $\angle MAC = \angle MCA = 10^\circ$. Determine $\angle BMC$.

Solution 1. [S. Sun] Construct equilateral triangle MDC with M and D on opposite sides of AC and equilateral triangle AME with M and Z on opposite sides of AB. Since $AM = MC$, these equilateral triangles are congruent. Since $AM = MD$ and

$$\angle AMD = \angle AMC - \angle DMC = 160^\circ - 60^\circ = 100^\circ \ ,$$

$\angle MAD = \angle MDA = 40^\circ$. Since $ME = AM = MC$, triangle EMC is isosceles. Since

$$\angle EMC = 360^\circ - \angle EMA - \angle AMC = 360^\circ - 60^\circ - 160^\circ = 140^\circ \ ,$$

$\angle EMC = \angle MCE = 20^\circ$. As $\angle MCB = 20^\circ = \angle MCE$, E, B, C are collinear. Now

$$\begin{aligned} \angle EBA &= \angle BAC + \angle BCA = 20^\circ + 30^\circ = 50^\circ \\ &= 60^\circ - 10^\circ = \angle EAM - \angle BAM = \angle EAB \ , \end{aligned}$$

so that $BE = AE = ME$ and triangle BEM is isosceles. Since $\angle BEM = \angle BEA - \angle MEA = 80^\circ - 60^\circ = 20^\circ$, it follows that

$$\angle BMC = 360^\circ - \angle EMB - \angle EMA - \angle AMC = 360^\circ - 80^\circ - 60^\circ - 160^\circ = 60^\circ \ .$$

Solution 2. Let O be the circumcentre of the triangle BAC; this lies on the opposite side of AC to B. Since the angle subtended at the centre by a chord is double that subtended at the circumference, we have that

$$\angle AOC = 2(180^\circ - \angle ABC) = 2(180^\circ - 130^\circ) = 100^\circ \ .$$

The right bisector of the segment AC passes through the apex of the isosceles triangle MAC and the centre O of the circumcircle of triangle BAC. We have that $\angle AOM = 50^\circ$, $\angle AMO = \frac{1}{2}\angle AMC = 80^\circ$, and

$$\angle MAO = 180^\circ - 50^\circ - 80^\circ = 50^\circ \ .$$

Therefore, triangle MAO is isosceles with $MA = MO$.

Observe that $\angle BAO = \angle BAC + \angle MAO - \angle MAC = 60^\circ$ and that $AO = BO$, so that triangle BAO is equilateral and so $BA = BO$. Since B and M are both equidistant from A and O, the line BM must right bisect the segment AO at N, say. Therefore, $\angle MNO = 90^\circ$, so that $\angle NMO = 40^\circ$. It follows that

$$\angle BMC = 180^\circ - \angle CMO - \angle NMO = 180^\circ - 80^\circ - 40^\circ = 60^\circ \ .$$

Solution 3. [M. Essafty] Let $\alpha = \angle MBA$, so that $\angle MBC = 130^\circ - \alpha$. From the trigonometric version of Ceva's Theorem, we have that

$$\sin\alpha \sin 20^\circ \sin 10^\circ = \sin(130^\circ - \alpha)\sin 10^\circ \sin 10^\circ$$

$$\Rightarrow 2\sin alpha \sin 10^\circ \cos 10^\circ = \sin(130^\circ - \alpha)\sin 10^\circ$$

$$\Rightarrow 2\sin\alpha\cos 10^\circ = \cos(40^\circ - \alpha) = \cos 40^\circ \cos\alpha + \sin 40^\circ \sin\alpha \ .$$

Dividing both sides by $\cos 40^\circ \cos\alpha$ yields that

$$2\cos\alpha\left(\frac{2\cos 10^\circ}{\cos 40^\circ} - \frac{\sin 40^\circ}{\cos 40^\circ}\right) = 1 \ .$$

Therefore

$$\begin{aligned}\cot\alpha &= \frac{\cos 10^\circ + \cos 10^\circ - \cos 50^\circ}{\cos 40^\circ} \\ &= \frac{\cos 10^\circ + 2\sin 30^\circ \sin 20^\circ}{\cos 40^\circ} \\ &= \frac{\cos 10^\circ + \sin 20^\circ}{\cos 40^\circ} = \frac{\cos 10^\circ + \cos 70^\circ}{\cos 40^\circ} \\ &= \frac{2\cos 40^\circ \cos 30^\circ}{\cos 40^\circ} = 2\cos 30^\circ = \sqrt{3}\ .\end{aligned}$$

Therefore $\alpha = 30^\circ$.

650. Suppose that the nonzero real numbers satisfy

$$\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = \frac{1}{xyz}\ .$$

Determine the minimum value of

$$\frac{x4}{x2 + y2} + \frac{y4}{y2 + z2} + \frac{z4}{z2 + x2}\ .$$

Solution 1. [W. Fu] Let $f(x, y, z)$ denote the expression

$$\frac{x4}{x2 + y2} + \frac{y4}{y2 + z2} + \frac{z4}{z2 + x2}\ .$$

Then

$$\begin{aligned}f(x,y,z) - f(x,z,y) &= \left(\frac{x4}{x2+y2} + \frac{y4}{y2+z2} + \frac{z4}{z2+x2}\right) - \left(\frac{x4}{x2+z2} + \frac{z4}{z2+y2} + \frac{y4}{y2+x2}\right) \\ &= \frac{x4 - y4}{x2+y2} + \frac{y4 - z4}{y2+z2} + \frac{z4 - x4}{z2+x2} \\ &= (x2 - y2) + (y2 - z2) + (z2 - x2) = 0\ .\end{aligned}$$

Thus, $f(x, y, z) = f(x, z, y)$ and

$$\begin{aligned}f(x,y,z) &= \frac{1}{2}(f(x,y,z) + f(x,z,y)) \\ &= \frac{1}{2}\left[\frac{x4+y4}{x2+y2} + \frac{y4+z4}{y2+z2} + \frac{z4+x4}{z2+x2}\right] \\ &= \frac{1}{2}\left[\left(x2 + y2 - \frac{2x^2y2}{x2+y2}\right) + \left(y2 + z2 - \frac{2y^2z2}{y2+z2}\right) + \left(z2 + x2 - \frac{2z^2x2}{z2+x2}\right)\right] \\ &= (x2 + y2 + z2) - \frac{1}{2}\left(\frac{2x^2y2}{x2+y2} + \frac{2y^2z2}{y2+z2} + \frac{2z^2x2}{z2+x2}\right)\end{aligned}$$

Observe that

$$x2 + y2 + z2 = \frac{1}{2}[(x2 + y2) + (y2 + z2) + (z2 + x2)] \geq xy + yz + zx = 1$$

and that $2x^2y2 \leq x4 + y4$. Hence

$$\begin{aligned}f(x,y,z) &\geq 1 - \frac{1}{2}\left(\frac{x4+y4}{x2+y2} + \frac{y4+z4}{y2+z2} + \frac{x4+x4}{z2+x2}\right) \\ &= 1 - \frac{1}{2}[f(x,y,z) + f(x,z,y)] = 1 - f(x,y,z)\ ,\end{aligned}$$

from which $f(x, y, z) \geq \frac{1}{2}$. Equality occurs if and only if $x = y = z = 1/\sqrt{3}$.

Solution 2. [S. Sun] From the Arithmetic-Geometric Means Inequality, we have that

$$\frac{x4}{x2 + y2} + \frac{1}{4}(x2 + y2) \geq x2$$

with a similar inequality for the other pairs of variables. Adding the three inequalities obtained, we find that

$$\frac{x4}{x2 + y2} + \frac{y4}{y2 + z2} + \frac{z4}{z2 + x2} + \frac{1}{2}(x2 + y2 + z2) \geq x2 + y2 + z2$$

from which

$$\frac{x4}{x2 + y2} + \frac{y4}{y2 + z2} + \frac{z4}{z2 + x2} \geq \frac{1}{2}(x2 + y2 + z2) \ ,$$

with equality if and only if $x = y = z$. Since $(x - y)2 + (y - z)2 + (z - x)2 \geq 0$, it follows that $x2 + y2 + z2 \geq xy + yz + zx = 1$. Therefore

$$\frac{x4}{x2 + y2} + \frac{y4}{y2 + z2} + \frac{z4}{z2 + x2} \geq \frac{1}{2}$$

with equality if and only if $x = y = z = 1/\sqrt{3}$.

Solution 3. [K. Zhou; G. Ajjanagadde; M. Essafty] Since $(x - y)2 \geq 0$, *etc.*, we have that $x2 + y2 + z2 \geq xy + yz + zx$. By the Cauchy-Schwarz Inequality, we have that

$$\left[\left(\frac{x2}{\sqrt{x2 + y2}}\right)2 + \left(\frac{y2}{\sqrt{y2 + z2}}\right)2 + \left(\frac{z2}{\sqrt{z2 + x2}}\right)2\right][(\sqrt{x2 + y2})2 + (\sqrt{y2 + z2})2 + (\sqrt{z2 + x2})2]$$
$$\geq (x2 + y2 + z2)2 \ ,$$

whence

$$\left(\frac{x4}{x2 + y2} + \frac{y4}{y2 + z2} + \frac{z4}{z2 + x2}\right)[(x2 + y2) + (y2 + z2) + (z2 + x2)] \geq (x2 + y2 + z2)2 \ ,$$

so that

$$\frac{x4}{x2 + y2} + \frac{y4}{y2 + z2} + \frac{z4}{z2 + x2} \geq \frac{x2 + y2 + z2}{2} \geq \frac{xy + yz + zx}{2} = \frac{1}{2} \ .$$

Equality occurs when $x = y = z = 1/\sqrt{3}$.

Solution 4. Observe that the given condition is equivalent to $xy + yz + zx = 1$. Since the expression to be minimized is the same when (x, y, z) is replaced by $(-x, -y, -z)$ and since two of the variables must have the same sign, we may assume that x and y are both positive.

Suppose, first, that $z > 0$. Since $x2 + y2 \geq 2xy$, we have that

$$\frac{x4}{x2 + y2} = x2 - \frac{x^2y2}{x2 + y2} \geq x2 - \frac{xy}{2} \ ,$$

with similar inequalities for the other pairs of variables. Therefore, the expression to be minimized is not less that

$$(x2 + y2 + z2) - \frac{1}{2}(xy + yz + zx) \geq (xy + yz + zx) - \frac{1}{2}(xy + yz + zx) = \frac{1}{2} \ .$$

Equality occurs if and only if $x = y = z = 1/\sqrt{3}$.

Regardless of the signs of the variables, if the largest of $x2$, $y2$, $z2$ is at least 2, we show that the expression is not less that 1. For example, if $x2 \geq 2$, $x2 \geq y2$, we find that

$$\frac{x4}{x2 + y2} \geq \frac{x4}{2x2} = \frac{x2}{2} \geq 1 \ .$$

Henceforth, assume that $x2$, $y2$, $z2$ are less than 2 and that $z < 0$. Then $xy < 2$. Since $0 > z = (1 - xy)/(x + y)$, then $xy > 1$, so that $x + y \geq 2\sqrt{xy} > 2$. Hence

$$|z| = \frac{xy - 1}{x + y} \leq \frac{1}{2} \ .$$

If $x > y$, then (because $xy > 1$), $x > 1$, so that

$$\frac{x4}{x2 + y2} > \frac{x4}{2x2} > \frac{1}{2} \ .$$

If $y > z$, then $y > 1 > |z|$ and

$$\frac{y4}{y2 + z2} > \frac{y4}{2y2} > \frac{1}{2} \ .$$

In any case, when $z < 0$, the quantity to be minimized exceeds $1/2$. Therefore, the minimum value is $1/2$, achieved when $(x, y, z) = (3^{-1/2}, 3^{-1/2}, 3^{-1/2})$.

Solution 5. [B. Wu] We first establish a lemms: *if a, b, u, v are positive, then*

$$\frac{a2}{u} + \frac{b2}{v} \geq \frac{(a + b)2}{u + v}$$

with equality if and only if $a : u = b : v$. To see this, subtract the right side from the left to get a fraction whose numerator is $(av - bu)2$.

Applying this to the given expression yields that

$$\begin{aligned} \frac{(x2)2}{y2 + z2} + \frac{(y2)2}{z2 + x2} + \frac{(z2)2}{x2 + y2} & \\ \geq \frac{(x2 + y2 + z2)2}{2(x2 + y2 + z2)} &= \frac{x2 + y2 + z2}{2} \\ \geq \frac{xy + yz + zx}{2} &= \frac{1}{2} \ . \end{aligned}$$

Equality occurs if and only if $x = y = z = 1/\sqrt{3}$.

Solution 6. [M. Essafty] Squaring both sides of the equation $2x2 = (x2 + y2) + (x2 - y2)$ yields that

$$\begin{aligned} 4x4 &= (x2 + y2)2 + (x2 - y2)2 + 2(x2 + y2)(x2 - y2) \\ &\geq (x2 + y2)2 + 2(x2 + y2)(x2 - y2) \end{aligned}$$

whence

$$\frac{4x4}{x2 + y2} \geq 3x2 - y2 \ .$$

Taking account of similar inequalities for other pairs of variables, we obtain that

$$\frac{4x4}{x2 + y2} + \frac{4y4}{y2 + z2} + \frac{4z4}{z2 + x2} \geq 2(x2 + y2 + z2) \geq 2(xy + yz + zx) = 2 \ ,$$

from which we conclude that the minimum value is $\frac{1}{2}$. This is attained when $x = y = z = 1/\sqrt{3}$.

Solution 7. [O. Xia] Recall that, for $r > 0$, $r + (1/r) \geq 2$, so that $r \geq 2 - (1/r)$. It follows that

$$\begin{aligned} \frac{x4}{x2 + y2} &= \left(\frac{x2}{2}\right)\left(\frac{2x2}{x2 + y2}\right) \\ &\geq \left(\frac{x2}{2}\right)\left(2 - \frac{x2 + y2}{2x2}\right) \\ &= x2 - \frac{x2 + y2}{4} \end{aligned}$$

with similar equalities for the other two terms in the problem statement. Equality occurs if and only if $x2 = y2 = z2$.

Adding the three equalities yields that Determine the minimum value of

$$\frac{x4}{x2+y2} + \frac{y4}{y2+z2} + \frac{z4}{z2+x2} \geq \frac{x2+y2+z2}{2} \ .$$

As before, we see that the right member assumes its minimum value of $\frac{1}{2}$ when $x = y = z = 1/\sqrt{3}$.

651. Determine polynomials $a(t)$, $b(t)$, $c(t)$ with integer coefficients such that the equation $y2+2y = x3-x2-x$ is satisfied by $(x, y) = (a(t)/c(t), b(t)/c(t))$.

Solution. The equation can be rewritten $(y+1)2 = (x-1)2(x+1)$. Let $x+1 = t2$ so that $y+1 = (t2-2)t$. Thus, we obtain the solution

$$(x, y) = (t2 - 1, t3 - 2t - 1) \ .$$

With these polynomials, both sides of the equation are equal to $t6 - 4t4 + 4t2 - 1$.

652. (a) Let m be any positive integer greater than 2, such that $x2 \equiv 1 \pmod m$ whenever the greatest common divisor of x and m is equal to 1. An example is $m = 12$. Suppose that n is a positive integer for which $n + 1$ is a multiple of m. Prove that the sum of all of the divisors of n is divisible by m.

(b) Does the result in (a) hold when $m = 2$?

(c) Find all possible values of m that satisfy the condition in (a).

(a) *Solution 1.* Let $n + 1$ be a multiple of m. Then $\gcd(m, n) = 1$. We observe that n cannot be a square. Suppose, if possible, that $n = r2$. Then $\gcd(r, m) = 1$. Hence $r2 \equiv 1 \pmod m$. But $r2 + 1 \equiv 0 \pmod m$ by hypothesis, so that 2 is a multiple of m, a contradiction.

As a result, if d is a divisor of n, then n/d is a distinct divisor of n. Suppose $d|n$ (read "d divides n"). Since m divides $n + 1$, therefore $\gcd(m, n) = \gcd(d, m) = 1$, so that $d2 = 1 + bm$ for some integer b. Also $n + 1 = cm$ for some integer c. Hence

$$d + \frac{n}{d} = \frac{d2 + n}{d} = \frac{1 + bm + cm - 1}{d} = \frac{(b + c)m}{d} \ .$$

Since $\gcd(d, m) = 1$ and $d + n/d$ is an integer, d divides $b + c$ and so $d + n/d \equiv 0 \pmod m$.

Hence

$$\sum_{d|n} d = \sum\{(d + n/d) : d|n, d < \sqrt{n}\} \equiv 0 \pmod m$$

as desired.

Solution 2. Suppose that $m > 1$ and m divides $n + 1$. Then gcd $(m, n) = 1$. Suppose, if possible, that $n = r2$ for some r. Then, since gcd $(m, r) = 1$, $r2 \equiv 1 \pmod r$. Therefore m divides both $r2 + 1$ and $r2 - 1$, so that $m = 2$. But this gives a contradiction. Hence n is not a perfect square.

Suppose that d is a divisor of n. Then the greatest common divisor of m and d is 1, so that $d2 \equiv 1 \pmod n$. Suppose that $de = n$. Then $e \neq 1d$ and the greatest common divisor of m and e is 1. Therefore, there are numbers u and v for which both du and ev are congruent to 1 modulo m. Since $n \equiv -1$ and $d2 \equiv 1 \pmod m$, it follows that

$$d + e \equiv d + un \equiv u(d2 + n) \equiv u(1 - 1) = 0$$

mod m), from which it can be deduced that m divides the sum of all the divisors of n.

Solution 3. Suppose that $n+1 \equiv 0 \pmod{m}$. As in the first solution, it can be established that n is not a perfect square. Let x be any positive divisor of n and suppose that $xy = n$; x and y are distinct. Since gcd $(x, m) = 1$, $x2 \equiv 1 \pmod{m}$, so that

$$y = x2y \equiv xn \equiv -x \pmod{m}$$

whence $x + y$ is a multiple of m. Thus, the divisors of n comes in pairs, each of which has sum divisible by m, and the result follows.

Solution 4. [M. Boase] As in the second solution, if $xy = n$, then $x2 \equiv y2 \equiv 1 \pmod{m}$ so that

$$0 \equiv x2 - y2 \equiv (x-y)(x+y) \pmod{m}.$$

For any divisor r of m, we have that

$$x(x-y) \equiv x2 - xy \equiv 2 \pmod{r}$$

from which it follows that the greatest common divisor of m and $x - y$ is 1. Therefore, m must divide $x + y$ and the solution can be completed as before.

(b) *Solution.* When $m = 2$, the result does not hold. The hypothesis is true. However, the conclusion fails when $n = 9$ since $9 + 1$ is a multiple of 2, but $1 + 3 + 9 = 13$ is odd.

(c) *Solution 1.* By inspection, we find that $m = 1, 2, 3, 4, 6, 8, 12, 24$ all satisfy the condition in (a).

Suppose that m is odd. Then $\gcd(2, m) = 1 \Rightarrow 22 = 4 \equiv 1 \pmod{m} \Rightarrow m = 1, 3$.

Suppose that m is not divisible by 3. Then $\gcd(3, m) = 1 \Rightarrow 9 = 32 \equiv 1 \pmod{m} \Rightarrow m = 1, 2, 4, 8$. Hence any further values of m not listed in the above must be even multiples of 3, that is, multiples of 6.

Suppose that $m \geq 30$. Then, since $25 = 52 \not\equiv 1 \pmod{m}$, m must be a multiple of 5.

It remains to show that in fact m cannot be a multiple of 5. We observe that there are infinitely many primes congruent to 2 or 3 modulo 5. [To see this, let $q_1, \cdots, q_s$ be the s smallest odd primes of this form and let $Q = 5q_1 \cdots q_s + 2$. Then Q is odd. Also, Q cannot be a product only of primes congruent to ± 1 modulo 5, for then Q itself would be congruent to ± 1. Hence Q has an odd prime factor congruent to ± 2 modulo 5, which must be distinct from $q_1, \cdots, q_s$. Hence, no matter how many primes we have of the desired form, we can always find one more.] If possible, let m be a multiple of 5 with the stated property and let q be a prime exceeding m congruent to ± 2 modulo 5. Then $\gcd(q, m) = 1 \Rightarrow q2 \equiv 1 \pmod{m} \Rightarrow q2 \equiv 1 \pmod{5}$ $\Rightarrow q \not\equiv \pm 2 \pmod{5}$, yielding a contradiction. Thus, we have given a complete collection of suitable numbers m.

Solution 2. [J. Rickards] Suppose that a suitable value of m is equal to a power of 2, Then $32 \equiv 1$ (mod m) implies that m must be equal to 4 or 8. It can be checked that both these values work.

Suppose that $m = p^a q$, where p is an odd prime and p and q are coprime. By the Chinese Remainder Theorem, there is a value of x for which $x \equiv 1 \pmod{q}$ and $x \equiv 2 \pmod{p^a}$. Then $x2 \equiv 1 \pmod{m}$, so that $4 \equiv x2 \equiv 1 \pmod{p^a}$ and thus p must equal 3. Therefore, m must be divisible by only the primes 2 and 3. Therefore $25 = 52 \equiv 1 \pmod{m}$, with the result that m must divide 24. Checking reveals that the only possibilities are $m = 3, 4, 6, 8, 12, 24$.

Solution 3. [D. Arthur] Suppose that $m = ab$ satisfies the condition of part (a), where the greatest common divisor of a and b is 1. Let gcd $(x, a) = 1$. Since a and b are coprime, there exists a number t such that $at \equiv 1 - x \pmod{b}$, so that $z = x + at$ and b are coprime. Hence, the greatest common divisor of z and ab equals 1, so that $z2 \equiv 1 \pmod{ab}$, whence $x2 \equiv z2 \equiv 1 \pmod{a}$. Thus a (and also b) satisfies the condition of part (a).

When m is odd and exceeds 3, then gcd $(2, m) = 1$, but $22 = 4 \not\equiv 1 \pmod{m}$, so m does not satisfy the condition. When $m = 2^k$ for $k \geq 4$, then gcd $(3, m) = 1$, but $32 = 9 \not\equiv 1 \pmod{m}$. It follows from the first paragraph that if m satisfies the condition, it cannot be divisible by a power of 2 exceeding 8 nor by an odd number exceeding 3. This leaves the possibilities $1, 2, 3, 4, 6, 8, 12, 24$, all of which satisfy the condition.

653. Let $f(1) = 1$ and $f(2) = 3$. Suppose that, for $n \geq 3$, $f(n) = \max\{f(r) + f(n-r) : 1 \leq r \leq n-1\}$. Determine necessary and sufficient conditions on the pair (a, b) that $f(a + b) = f(a) + f(b)$.

Solution 1. From the first few values of $f(n)$, we conjecture that $f(2k) = 3k$ and $f(2k+1) = 3k+1$ for each positive integer k. We establish this by induction. It is easily checked for $k = 1$. Suppose that it holds up to $k = m$.

Suppose that $2m+2$ is the sum of two positive even numbers $2x$ and $2y$. Then $f(2x)+f(2y) = 3(x+y) = 3(m+1)$. If $2m+2$ is the sum of two positive odd numbers $2u+1$ and $2v+1$, then

$$f(2u+1) + f(2v+1) = (3u+1) + (3v+1) = 3(u+v) + 2 < 3(u+v+1) = 3(m+1) \ .$$

Hence $f(2(m+1)) = 3(m+1)$.

Suppose $2m+3$ is the sum of $2z$ and $2w+1$. Then $z + w = m+1$ and

$$f(2z) + f(2w+1) = 3z + 3w + 1 = 3(z+w) + 1 = 3(m+1) + 1 \ .$$

Hence $f(2(m+1)+1) = 3(m+1)+1$. The conjecture is established by induction.

By checking cases on the parity of a and b, one verifies that $f(a+b) = f(a)+f(b)$ if and only if at least one of a and b is even. (If a and b are both odd, the left side is divisible by 3 while the right side is not.)

Solution 2. [K. Yeats] By inspection, we conjecture that $f(n+1) = f(n)+2$ when n is odd, and $f(n+1) = f(n)+1$ when n is even. This is true for $n = 1, 2$. Suppose it holds up to $n = 2k$. If $2k+1 = i+j$ with i even and j odd, then $f(i-1)+f(j+1) = f(i)-2+f(j)+2 = f(i)+f(j)$ and $f(i+1)+f(j-1) = f(i)+1+f(j)-1 = f(i)+f(j)$ (where defined), so in particular $f(2k+1) = f(2k)+f(1) = f(2k)+1$. Note that this also tells us that $f(2k+1) = f(i)+f(j)$ whenever $i+j = 2k+1$. Now consider $2k+2 = i+j$. If i and j are both even, then

$$f(i+1) + f(j-1) = f(i) + 1 - f(j) - 2 = f(i) + f(j) - 1$$

while if i and j are both odd, then

$$f(i+1) + f(j-1) = f(i) + 2 - f(j) - 1 = f(i) + f(j) + 1 \ .$$

Thus, $f(2k+2) = f(i)+f(j)$ if and only if i and j are both even. In particular, $f(2k+2) = f(2k)+f(2) = f(2k+1)-1+3 = f(2k)+2$. We thus find that $f(a+b) = f(a)+f(b)$ if and only if at least one of a and b is even.