Solutions for March

605. Prove that the number $299\cdots 998200\cdots 029$ can be written as the sum of three perfect squares of three consecutive numbers, where there are $n-1$ nines between the first 2 and the 8, and $n-1$ zeros between the last pair of twos.

Solution. Let $a-1$, a, $a+1$ be the three consecutive numbers. The sum of their square is $3a^2+2$; setting this equal to the given number yields

$$\begin{aligned} a^2 &= 9\cdot 10^{2n+1} + \cdots + 9\cdot 10^{n+3} + 9\cdot 10^{n+2} + 4\cdot 10^{n+1} + 9 \\ &= (10^n-1)10^{n+2} + 4\cdot 10^{n+1} + 9 = 10^{2n+2} - 6\cdot 10^{n+1} + 9 \\ &= (10^{n+1}-3)^2 \ , \end{aligned}$$

so that $a = 10^{n+1}-3$.

606. Let $x_1 = 1$ and let $x_{n+1} = \sqrt{x_n + n^2}$ for each positive integer n. Prove that the sequence $\{x_n : n > 1\}$ consists solely of irrational numbers and calculate $\sum_{k=1}^n \lfloor x_k^2 \rfloor$, where $\lfloor x \rfloor$ is the largest integer that does not exceed x.

Solution. We prove that x_n is nonrational as well as positive for $n \geq 2$. Note that x_2 is nonrational. Suppose that $n \geq 2$ and that x_{n+1} were rational; then $x_n = x_{n+1}^2 - n^2$ would also be rational; repeating this would lead to x_2 being rational and a contradiction.

Observe that, for any positive integer $n \geq 2$,

$$x_n = \sqrt{x_{n-1} + (n-1)^2} > n-1 \ .$$

We prove by induction that $x_n < n$. This is true for $n = 2$. If $x_{n-1} < n-1$, then

$$x_n^2 = x_{n-1} + (n-1)^2 < (n-1)n < n^2 \ ,$$

and the desired result follows. Thus, for each $n \geq 2$, $\lfloor x_n \rfloor = n-1$,

For $n \geq 3$,

$$\lfloor x_n^2 \rfloor = \lfloor x_{n-1} + (n-1)^2 \rfloor = (n-2) + (n-1)^2 = n^2 - n - 1 = n(n-1) - 1 \ .$$

Therefore

$$\begin{aligned} \sum_{k=1}^n \lfloor x_k^2 \rfloor &= \lfloor x_1^2 \rfloor + \lfloor x_2^2 \rfloor + \sum_{k=3}^n \lfloor x_k^2 \rfloor \\ &= 3 + \left[(\sum_{k=3}^n k(k-1) \right] - (n-2) \\ &= 5 - n + \frac{1}{3} \sum_{k=3}^n [(k+1)k(k-1) - k(k-1)(k-2)] \\ &= 5 - n + \frac{1}{3}[(n+1)n(n-1) - 6] = 3 - n + \frac{1}{3}(n^3 - n) \\ &= \frac{1}{3}(n^3 - 4n + 9) \ , \end{aligned}$$

607. Solve the equation

$$\sin x \left(1 + \tan x \tan \frac{x}{2} \right) = 4 - \cot x \ .$$

Solution. For the equation to be defined, x cannot be a multiple of π, so that $\sin x \neq 0$. Rearranging the terms of the equation and manipulating yields that

$$\begin{aligned} 4 &= \cot x + \sin x \left(\frac{\cos x \cos \frac{x}{2} + \sin x \sin \frac{x}{2}}{\cos x \cos \frac{x}{2}} \right) \\ &= \cot x + \sin x \left(\frac{\cos(x - (x/2))}{\cos x \cos(x/2)} \right) \\ &= \frac{\cos x}{\sin x} + \frac{\sin x}{\cos x} \\ &= \frac{\cos^2 x + \sin^2 x}{\sin x \cos x} = \frac{2}{\sin 2x} \ , \end{aligned}$$

whence $\sin 2x = \frac{1}{2}$. Therefore $x = (-1)^k \frac{\pi}{12} + \frac{k\pi}{2}$, where k is an integer.

608. Find all positive integers n for which n, $n^2 + 1$ and $n^3 + 3$ are simultaneously prime.

Solution. If $n = 2$, then the numbers are 2, 5 and 11 and all are prime. Otherwise, n must be odd. But in this case, the other two numbers are even exceeding 2 and so nonprime. Therefore $n = 2$ is the only possibility.

609. The first term of an arithmetic progression is 1 and the sum of the first nine terms is equal to 369. The first and ninth terms of the arithmetic progression coincide respectively with the first and ninth terms of a geometric progression. Find the sum of the first twenty terms of the geometric progression.

Solution. The sum of the first nine terms of an arithmetic progression is equal to 9/2 the sum of the first and ninth terms, from which it is seen that the ninth term is 81. Let r be the common ratio of the geometric progression whose first term is 1 and whose ninth term is 81. Then $r^8 = 81$, whence $r = \pm\sqrt{3}$. The sum of the first twenty terms of the geometric progression is $\frac{1}{2}(3^{10} - 1)(\pm\sqrt{3} + 1)$.

610. Solve the system of equations

$$\begin{aligned} \log_{10}(x^3 - x^2) &= \log_5 y^2 \\ \log_{10}(y^3 - y^2) &= \log_5 z^2 \\ \log_{10}(z^3 - z^2) &= \log_5 x^2 \end{aligned}$$

where $x, y, z > 1$.

Solution. For $x > 1$, let

$$f(x) = 5^{\log_{10}(x^3 - x^2)} \ .$$

The three equations are $f(x) = y^2$, $f(y) = z^2$ and $f(z) = x^2$. Since $x^3 - x^2 = x^2(x - 1)$ is increasing, f is an increasing function. If, say, $x < y$, then $y < z$ and $z < x$, yielding a contradiction. Thus, we can only have that $x = y = z$ and so

$$\log_{10}(x^3 - x^2) = \log_5 x^2 \ .$$

Let $2t = \log_5 x^2$ so that $t > 0$, $x^2 = 5^{2t}$ and so $x = 5^t$. Therefore

$$5^{3t} - 5^{2t} = 10^{2t} \Longrightarrow 5^t - 1 = 4^t \Longrightarrow 5^t - 4^t = 1 \ .$$

Since $5^t - 4^t = 4^t[(5/4)^t - 1]$ is an increasing function of t, we see that the equation for t has a unique solution, namely $t = 1$. Therefore $x = 5$.

611. The triangle ABC is isosceles with $AB = AC$ and I and O are the respective centres of its inscribed and circumscribed circles. If D is a point on AC for which $ID \| AB$, prove that $CI \perp OD$.

Solution. Since ABC is isosceles, the points A, O, I lie on the right bisector of BC. Let AO meet BC at P, DI meet BC at E, DO meet BC at F and CI meet DF at Q.

Suppose that angle A is less than $60°$. Then O lies between I and A, and Q lies within triangle APB. Since $DE \| AB$ and O is the centre of the circumcircle of ABC, we have that

$$\angle CDI = \angle BAC = \angle COI \ ,$$

so that $CIOD$ is concyclic. Therefore

$$\begin{aligned}\angle CQD &= 180° - (\angle QOI + \angle QIO) = 180° - (\angle ICD + \angle PIC) \\ &= 180° - (\angle ICP + \angle PIC) = 90° \ .\end{aligned}$$

Suppose that angle A exceeds $60°$. Then I lies between O and A, and Q lies on the same side of AP as C. Since

$$\angle IDC + \angle IOC = \angle BAC + \angle AOC = 180° \ ,$$

the quadrilateral $IOCD$ is concyclic. Therefore

$$\begin{aligned}\angle CQD &= 180° - (\angle DCQ + \angle QDC) = 180° - (\angle QCP + \angle ODC) \\ &= 180° - (\angle QCP + \angle OIC) = 180° - (\angle ICP + \angle PIC) = 90° \ .\end{aligned}$$

Finally, if $\angle A = 60°$, then I and O coincide so that $DF = DE \| AB$ and the result is clear.