Solutions for November

577. $ABCDEF$ is a regular hexagon of area 1. Determine the area of the region inside the hexagon thst belongs to none of the triangles ABC, BCD, CDE, DEF, EFA and FAB.

Solution 1. Let O be the centre of the hexagon. The hexagon is the union of three nonoverlapping congruent rhombi, $ABCO$, $CDEO$, $EFAO$, each of area $\frac{1}{3}$. Each rhombus is the union of two congruent triangles, each of area $\frac{1}{6}$. In particular, $[ABC] = \frac{1}{2}$.

Let BD and AC intersect at P, and BF and AC at Q. By reflection about BE, we see that $BP = BQ$, triangle BPQ is equilateral and $\angle BPQ = 60°$. Since triangle BPC is isosceles (use symmetry) and $\angle BPC = 120°$, $CP = PB = PQ = BQ = QA$. Therefore $[BPC] = [BPQ] = [BQA] = \frac{1}{3}[ABC] = \frac{1}{18}$.

The union of triangles ABC, BCD, CDE, DEF, EFA, FAB is comprised of twelve nonoverlapping triangles congruent to either of the triangles BPC or BPQ, as so has area $\frac{2}{3}$. Therefore the area of the prescribed region inside the hexagon is $\frac{1}{3}$.

Solution 2. Let O be the centre of the hexagon. Since triangle ACE is the union of triangles OAC, OCE, OEA, and since $[OAC] = [BAC]$, $[OCE] = [DCE]$, $[OEA] = [FEA]$, it follows that $[ACE] = \frac{1}{2}[ABCDEF] = \frac{1}{2}$. As in Solution 1, we determine that $[BPQ] = [DUT] = [FRS] = \frac{1}{18}$, where $U = BD \cap CE$, $T = CE \cap DF$, $S = DF \cap EA$, $R = AE \cap BF$. Hence the area of the inner region is $\frac{1}{2} - (3 \times \frac{1}{18}) = \frac{1}{3}$.

Comment. In the original statement of the problem, triangle DEF was omitted by mistake from the statement. In this case, the region whose area was to be found is the union of $PQRSTU$ and one of the twelve small triangles; the answer is $1/3 + 1/18 = 7/18$.

578. $ABEF$ is a parallelogram; C is a point on the diagonal AE and D a point on the diagonal BF for which $CD \| AB$. The sements CF and EB intersect at P; the segments ED and AF intersect at Q. Prove that $PQ \| AB$.

Solution. Consider the shear that fixes A and B and shifts E in a parallel direction to E' so that $E'B \perp AB$. This shear preserves parallelism and takes $F \rightarrow F'$, $C \rightarrow C'$, $D \rightarrow D'$, $P \rightarrow P'$, $Q \rightarrow Q'$, so that $ABE'F'$ is a rectangle. A reflection about the right bisector of AB takes $E' \leftrightarrow F'$, $C' \leftrightarrow D'$, and so $P" \leftrightarrow Q'$. Hence $PQ \| P'Q' \| AB$.

579. Solve, for real x, y, z the equation

$$\frac{y^2 + z^2 - x^2}{2yz} + \frac{z^2 + x^2 - y^2}{2zx} + \frac{x^2 + y^2 - z^2}{2xy} = 1 \ .$$

Solution 1. Note that none of x, y, z can vanish. We have that

$$\begin{aligned} 0 = \frac{y^2 + z^2 - x^2}{2yz} &= \frac{z^2 + x^2 - y^2}{2zx} + \frac{x^2 + y^2 - z^2}{2xy} - 1 \\ &= \frac{xy^2 + xz^2 - x^3 + yz^2 + x^2y - y^3 + x^2z + y^2z - z^3 - 2xyz}{2xyz} \\ &= \frac{(x + y - z)(xy + z^2) + (x^2 + y^2 - xy)z - (x^2 + y^2 - xy)(x + y)}{2xyz} \\ &= \frac{(x + y - z)(z^2 - (x - y)^2)}{2xyz} = \frac{(x + y - z)(z + x - y)(z + y - x)}{2xyz} \ , \end{aligned}$$

whereupon (x, y, z) is a solution if and only if one of the conditions $x + y = z$, $y + z = x$ and $z + x = y$ is satisfied.

Solution 2. We must have $xyz \neq 0$ for the equation to be defined. Suppose that a, b, c are such that $y^2 + z^2 - x^2 = 2ayz$, $z^2 + x^2 - y^2 = 2bzx$, $x^2 + y^2 - z^2 = 2cxy$. Then $a + b + c = 1$. Adding pairs of the three equations yields that

$$2z^2 = 2z(ay + bx) \ ,$$
$$2y^2 = 2y(az + cx) \ ,$$
$$2x^2 = 2x(bz + cy) \ .$$

Hence

$$bx + ay - z = 0 \ ,$$
$$cx - y + az = 0 \ ,$$
$$-x + cy + bz = 0 \ .$$

From the first two equations, we find that

$$x : y : z = (a^2 - 1) : (-c - ab) : (-b - ac) \ .$$

Plugging this into the third equation yields that

$$1 - a^2 - c^2 - abc - b^2 - abc = 0 \Longrightarrow a^2 + b^2 + c^2 = 1 - 2abc$$

$$\Longrightarrow 1 - 2(ab + bc + ca) = (a + b + c)^2 - 2(ab + bc + bc) = 1 - 2abc$$

$$\Longrightarrow ab + bc + ca = abc \Longrightarrow \frac{1}{a} + \frac{1}{b} + \frac{1}{c} = 1 \ ,$$

the last implication holding only if a, b, c are all nonzero.

But if any of a, b, c vanish, then two of them must vanish. Suppose that $a = b = 0$, $c = 1$. Then $z^2 = x^2 - y^2 = y^2 - x^2 = (x - y)^2$. This is impossible as $z \neq 0$.

Therefore

$$\frac{a + b}{ab} = \frac{1}{a} + \frac{1}{b} = 1 - \frac{1}{c} = \frac{c - 1}{c} = \frac{-(a + b)}{c} \ .$$

Therefore, either $a + b = 0$ or $ab = -c$. Similarly, either $b + c = 0$ or $bc = -a$, and either $c + a = 0$ or $ca = -b$. It is not possible for all of $a + b = 0$, $b + c = 0$ and $c + a = 0$ to occur.

Suppose wolog, $ab = -c$. If $b + c = 0$, then $a = 1$ and $ac = -b$. The condition $a = 1$ implies that $x^2 = (y - z)^2$, whence either $x + y = z$ or $x + z = y$ (which leads to $c = 1$ or $b = 1$).

If $ab = -c$, $bc = -a$, $ca = -b$, then $(abc)^2 = -abc$, so that $a^2 = b^2 = c^2 = -abc = 1$, whence $(a, b, c) = (1, 1, -1), (1, -1, 1), (-1, 1, 1)$.

In any case, two of a, b, c equal 1 and one of them equals -1. If, say $(a, b, c) = (1, 1, -1)$, then $x^2 - (y - z)^2 = y^2 - (z - x)^2 = z^2 - (x + y)^2 = 0$, whence

$$0 = (x - y + z)(x + y - z) = (y - z + x)(y + z - x) = (z - x - y)(x + y + z) \ .$$

The solutions $x + y + z = 0$ is not possible; otherwise

$$\frac{y^2 + z^2 - x^2}{2yz} + \frac{z^2 + x^2 - y^2}{2zx} + \frac{x^2 + y^2 - z^2}{2xy} = \frac{-2yz}{2yz} + \frac{-2zx}{2zx} + \frac{-2xy}{2xy} = -3 \ .$$

Therefore $x + y - z = 0$. Similarly, if $(a, b, c) = (1, -1, 1)$, then $z + x - y = 0$, and if $(a, b, c) = (-1, 1, 1)$, then $y + z - x$. it is readily checked that these solutions work.

580. Two numbers m and n are two perfect squares with four decimal digits. Each digit of m is obtained by increasing the corresponding digit of n be a fixed positive integer d. What are the possible values of the pair (m, n).

Solution. Let

$$n = y^2 = p \times 10^3 + q \times 10^2 + r \times 10 + s$$

and

$$m = x^2 = (p+d) \times 10^3 + (q+d) \times 10^2 + (r+d) \times 10 + (s+d) \ ,$$

where $1 \le p < p+d \le 9$, $0 \le q < q+d \le 9$, $0 \le r < r+d \le 9$, $0 \le s < s+d \le 9$. Then

$$(x+y)(x-y) = x^2 - y^2 = d \times 1111 = d \times 11 \times 101 \ .$$

Since $10^3 \le n < m < 10^4$, $32 \le y < x \le 99$, it follows that $x+y < 200$ and $x-y \le 67$. Since the prime 101 must be a factor of either $x+y$ or $x-y$ and since each multiple of 101 exceeds 200, we must have that $x+y = 101$ and $x-y = 11d$. Since x and y must have opposite parity, d must be odd.

Since $64 \le 2y = 101 - 11d$, $11d \le 37$, so that $d \le 3$. Therefore, either $d = 1$ or $d = 3$. The case $d = 1$ leads to $x+y = 101$ and $x-y = 11$, so that $(x,y) = (56,45)$ and $(m,n) = (3136,2025)$. The case $d = 3$ leads to $x+y = 101$ and $x-y = 33$, so that $(x,y) = (67,34)$ and $(m,n) = (4489,1156)$.

Thus, there are two possibilities for (m,n): $(3136,2025)$, $(4489,1156)$.

581. Let $n \ge 4$. The integers from 1 to n inclusive are arranged in some order around a circle. A pair (a,b) is called *acceptable* if $a < b$, a and b are not in adjacent positions around the circle and at least one of the arcs joining a and b contains only numbers that are less than both a and b. Prove that the number of acceptable pairs is equal to $n-3$.

Solution 1. We prove the result by induction. Let $n = 4$. If 2 and 4 are not adjacent, then $(2,4)$ is acceptable. If 2 and 4 are adjacent, then 1 must be between 3 and one of 2 and 4, in which case $(2,3)$ or $(3,4)$ is the only acceptable pair.

Suppose that $n \ge 5$, that the result holds for $n-1$ numbers and that a configuration of the numbers 1 to n, inclusive is given. The number 1 must lie between two immediate neighbours u and v that are non-adjacent. Thus, the pair (u,v) is acceptable.

Now remove the number 1 and replace each remaining number r by $r' = r-1$ to obtain a configuration of $n-1$ numbers. We show that (r',s') is acceptable in the latter configuration if and only if (r,s) is acceptable in the given configuration.

If (r',s') is acceptable, then r' and s' are not adjacent and there is an arc of smaller numbers between them. The addition of 1 to these numbers and the insertion of 1 will not change either characteristic for (r,s). On the other hand, if $(r,s) \ne (u,v)$ is acceptable in the original configuration , then r and s are not adjacent and each arc connecting them must contain some number other than 1; one of these arcs, at least, contains only numbers less than both r and s. In the final configuration, r' and s' continue to be non-adjacent and a corresponding arc contains only numbers less than both of them.

By the induction hypothesis, there are $(n-1)-3 = n-4$ acceptable pairs in the latter configuration, and so, with the inclusion of (u,v), there are $(n-4)+1 = n-3$ acceptable pairs in the given configuration.

Solution 2. We formulate the more general result that, if $n \ge 3$ and any n distinct real numbers are arranged in a circle and acceptability of pairs is defined as in the problem, then there are precisely $n-3$ acceptable pairs. This is equivalent to the given problem, since there is an order-preserving one-one mapping from these numbers to $\{1,2,\cdots,n\}$ that takes the kth largest of them to k.

We use induction. As in the previous solution, we see that it is true for $n = 3$ and $n = 4$. Let $n \ge 5$ and suppose that the largest three numbers are u, v, w. At least one of these three pairs is non-adjacent; otherwise, if w is adjacent to both u and v, then w is between u and v; since u and v are separated on both sides by at least one number, they are non-adjacent. This pair is acceptable, since a larger number can appear on at most one of the arcs connecting them.

Suppose that this acceptable pair is (u, v). Since all the numbers in at least one of the arcs connecting them are smaller, there is no acceptable pair (a, b) with a and b on different arcs joing u and v.

Consider two "circles" of numbers consisting of the $k \geq 3$ numbers of one arc A determined by (u, v) including u and v, and the $n + 2 - k$ numbers of the other arc B determined by (u, v) including u and v.

The set A contains exactly $k - 3$ acceptable pairs and the set B $n - 1 - k$ acceptable pairs, by the induction hypothesis. Each of these pairs is acceptable in the original circle of n numbers since none of the acceptable arcs includes u and v. Therefore, the original circle has $1+(k-3)+(n-1-k) = n-3$ acceptable pairs.

Solution 3. [C. Bruggeman] Suppose that $1 \leq k \leq n - 3$. Examine numbers counterclockwise from k until the first number a that exceeds k is reached; the examine numbers clockwise from k until the first number b that exceeds k is reached. Every number of the arc containing k between a and b is less than both a and b. Since there are at least three numbers exceeding k, at least one of them must be between a and b outside the arc containing k, so that a and b are not adjacent. Hence (a, b) is an acceptable pair.

We now prove that every acceptable pair is obtained exactly once in this way. Suppose that (a, b) is an acceptable pair with at least one of a and b not equal to $n - 1$ and n. Then, as one of the arcs between a and b must contain a number h bigger than at least one of them, the other arc must contain only numbers smaller than both of them. Let the largest such number be m. The m must engender the pair (a, b) by the foregoing process. Suppose that $k \leq n - 3$ is some other number other than a, b and m. Then m must lie on the arc between a and b opposite h between a and m or between m and b, or on the arc between a and b opposite m between a and h or between h and b; in each case, the pair engendered by k cannot be (a, b).

The only case remaining is $(n - 1, n)$ which may or may not be acceptable. If $(n - 1, n)$ is acceptable, then one arc connecting it must contain $n - 2$; by an argument similar to that in the last paragraph, no other element in this arc can engender $(n - 1, n)$. However, the largest element m in the other arc does not exceed $n - 3$ and it is the sole element that engenders $(n - 1, n)$.

Thus, there is a one-one correspondence between the numbers $1, 2, \cdots, n - 3$ and acceptable pairs; the desired result follows.

Comment. A. Abdi claims that the acceptable pair determine diagonals yielding a triangulation of the $n-$gon determined by the positions of the n numbers. Is this true?

582. Suppose that f is a real-valued function defined on the closed unit interval $[0, 1]$ for which $f(0) = f(1) = 0$ and $|f(x) - f(y)| < |x - y|$ when $0 \leq x < y \leq 1$. Prove that $|f(x) - f(y)| < \frac{1}{2}$ for all $x, y \in [0, 1]$. Can the number $\frac{1}{2}$ in the inequality be replaced by a smaller number and still result in a true proposition?

Solution 1. Suppose that $0 \leq x < y \leq 1$. If $y - x < \frac{1}{2}$, the result holds trivially. Suppose that $y - x \geq \frac{1}{2}$. Then

$$\begin{aligned} |f(y) - f(x)| &\leq |f(1) - f(y)| + |f(x) - f(0)| \\ &< (1 - y) + x = 1 - (y - x) \leq \frac{1}{2} \ , \end{aligned}$$

as desired.

The coefficient $\frac{1}{2}$ cannot be replaced by anything smaller. Suppose that $0 < \lambda < 1$; define

$$f_\lambda = \begin{cases} \lambda x & \text{if } 0 \leq x \leq \frac{1}{2} \\ \lambda(1 - x) & \text{if } \frac{1}{2} < x \leq 1 \ . \end{cases}$$

We show that f_λ has the desired property. However, note that $f_\lambda(\frac{1}{2}) - f_\lambda(0) = \frac{\lambda}{2}$, so that by our choice of λ, we can make the right side arbitrarily close to $\frac{1}{2}$.

If $0 \leq x < y \leq \frac{1}{2}$, then

$$|f(x) - f(y)| = \lambda|x - y| < |x - y| \ .$$

If $\frac{1}{2} \leq x < y \leq 1$, then

$$|f(x) - f(y)| = \lambda|(1-x) - (1-y)| = \lambda|x-y| < |x-y| \ .$$

Finally, suppose that $0 \leq x < \frac{1}{2} < y \leq 1$. Then

$$|f_\lambda(x) - f_\lambda(y)| = \lambda|x - (1-y)| = \lambda|(x+y) - 1| \ .$$

If $x + y \geq 1$, then

$$|(x+y) - 1| = (x+y) - 1 = (y-x) - (1-2x) < y - x \ ;$$

if $x + y \leq 1$, then

$$|(x+y) - 1| = 1 - (x+y) = (y-x) - (2y-1) < y - x \ .$$

In either case

$$|f_\lambda(x) - f_\lambda(y)| < \lambda(y-x) < y - x = |x-y| \ .$$

Solution 2. Since $|f(x) - f(y)| < |x-y|$, f must be continuous on $[0,1]$. [Provide an $\epsilon - \delta$ argument for this.] Therefore it assumes its maximum value M at a point $v \in [0,1]$ and its minimum value m at a point $u \in [0,1]$. We have that

$$0 \leq M = f(v) = f(v) - f(0) < v \leq 1$$

and

$$0 \geq m = f(u) = f(u) - f(0) > -u \geq 1 \ ,$$

since $|f(u) - f(0)| < |u - 0| = u$. Thus $|m| < u \leq 1$ and $M < v \leq 1$.

Suppose that $v < u$. Then

$$\begin{aligned} 2(M-m) &= M - m + (f(v) - f(u)) \\ &= f(v) + (f(1) - f(u)) + |f(u) - f(v)| \\ &< v + (1-u) + (u-v) = 1 \ . \end{aligned}$$

Suppose that $u < v$. Then

$$\begin{aligned} 2(M-m) &= M - m + (f(v) - f(u)) \\ &= |f(1) - f(v)| + |f(u)| + |f(v) - f(u)| \\ &< (1-v) + u + (v-u) = 1 \ . \end{aligned}$$

In either case, $M - m < \frac{1}{2}$. If $x, y \in [0,1]$, then $f(x)$ and $f(y)$ both lie in $[m, M]$ and so $|f(x) - f(y)| \leq M - m < \frac{1}{2}$.

583. Suppose that $ABCD$ is a convex quadrilateral, and that the respective midpoints of AB, BC, CD, DA are K, L, M, N. Let O be the intersection point of KM and LN. Thus $ABCD$ is partitioned into four quadrilaterals. Prove that the sum of the areas of two of these that do not have a common side is equal to the sum of the areas of the other two, to wit

$$[AKON] + [CMOL] = [BLOK] + [DNOM] \ .$$

Solution. Using the fact that triangles with equal bases and heights have the same area, we have that $[AKO] = [BKO]$, $[BLO] = [CLO]$, $[CMO] = [DMO]$ and $[DNO] = [ANO]$. Therefore

$$\begin{aligned} [AKON] + [CMOL] &= [AKO] + [ANO] + [CLO] + [CMO] \\ &= [BKO] + [BLO] + [DNO] + [DMO] = [BLOK] + [DNOM] \ . \end{aligned}$$