## PROBLEMS FOR SEPTEMBER

Please send your solutions to

Edward Barbeau
Department of Mathematics
University of Toronto
40 St. George Street
Toronto, ON M5S 2E4

no later than October 15, 2008. Electronic files can be sent to *barbeau@math.utoronto.ca*. It is important that your complete mailing address and your email address appear legibly on the front page. If you do not write your family name last, please underline it.

563. (a) Determine infinitely many triples $(a, b, c)$ of integers for which $a, b, c$ are not in arithmetic progression and $ab + 1$, $bc + 1$, $ca + 1$ are all squares.

(b) Determine infinitely many triples $(a, b, c)$ of integers for which $a, b, c$ are in arithemetic progression and $ab + 1$, $bc + 1$, $ca + 1$ are all squares.

(c) Determine infinitely many triples $(u, v, w)$ of integers for which $uv - 1$, $vw - 1$, $wu - 1$ are all squares. (Can it be arranged that $u, v, w$ are in arithmetic progression?)

564. Let $x_1 = 2$ and

$$x_{n+1} = \frac{2x_n}{3} + \frac{1}{3x_n}$$

for $n \geq 1$. Prove that, for all $n > 1$, $1 < x_n < 2$.

565. Let $ABC$ be an acute-angled triangle. Points $A_1$ and $A_2$ are located on side $BC$ so that the four points are ordered $B, A_1, A_2, C$; similarly $B_1$ and $B_2$ are on $CA$ in the order $C, B_1, B_2, A$ and $C_1$ and $C_2$ on side $AB$ in order $A, C_1, C_2, B$. All the angles $AA_1A_2$, $AA_2A_1$, $BB_1B_2$, $BB_2B_1$, $CC_1C_2$, $CC_2C_1$ are equal to $\theta$. Let $\mathfrak{T}_1$ be the triangle bounded by the lines $AA_1$, $BB_1$, $CC_1$ and $\mathfrak{T}_2$ the triangle bounded by the lines $AA_2$, $BB_2$, $CC_2$. Prove that all six vertices of the triangles are concyclic.

566. A deck of cards numbered 1 to $n$ (one card for each number) is arranged in random order and placed on the table. If the card numbered $k$ is on top, remove the $k$th card counted from the top and place it on top of the pile, not otherwise disturbing the order of the cards. Repeat the process. Prove that the card numbered 1 will eventually come to the top, and determine the maximum number of moves that is required to achieve this.

567. (a) Let $A, B, C, D$ be four distinct points in a straight line. For any points $X, Y$ on the line, let $XY$ denote the *directed* distance between them. In other words, a positive direction is selected on the line and $XY = \pm|XY|$ according as the direction $X$ to $Y$ is positive or negative. Define

$$(AC, BD) = \frac{AB/BC}{AD/DC} = \frac{AB \times CD}{BC \times DA} .$$

Prove that $(AB, CD) + (AC, BD) = 1$.

(b) In the situation of (a), suppose in addition that $(AC, BD) = -1$. Prove that

$$\frac{1}{AC} = \frac{1}{2}\left(\frac{1}{AB} + \frac{1}{AD}\right) ,$$

and that

$$OC^2 = OB \times OD ,$$

where $O$ is the midpoint of $AC$. Deduce from the latter that, if $Q$ is the midpoint of $BD$ and if the circles on diameters $AC$ and $BD$ intersect at $P$, $\angle OPQ = 90°$.

(c) Suppose that $A, B, C, D$ are four distinct on one line and that $P, Q, R, S$ are four distinct points on a second line. Suppose that $AP$, $BQ$, $CR$ and $DS$ all intersect in a common point $V$. Prove that $(AC, BD) = (PR, QS)$.

(d) Suppose that $ABQP$ is a quadrilateral in the plane with no two sides parallel. Let $AQ$ and $BP$ intersect in $U$, and let $AP$ and $BQ$ intersect in $V$. Suppose that $VU$ and $PQ$ produced meet $AB$ at $C$ and $D$ respectively, and that $VU$ meets $PQ$ at $W$. Prove that

$$(AB, CD) = (PQ, WD) = -1 \ .$$

568. Let $ABC$ be a triangle and the point $D$ on $AB$ be the foot of the altitude $AD$ from $A$. Suppose that $H$ lies on the segment $AD$ and that $BH$ and $CH$ intersect $AC$ and $AB$ at $E$ and $F$ respectively.

Prove that $\angle FDH = \angle HDE$.

569. Let $A, W, B, U, C, V$ be six points in this order on a circle such that $AU$, $BV$ and $CW$ all intersect in the common point $P$ at angles of $60°$. Prove that

$$|PA| + |PB| + |PC| = |PU| + |PV| + |PW| \ .$$