PROBLEMS FOR MARCH

Please send your solution to

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no later than March 31, 2008. It is important that your complete mailing address and your email address appear on the front page. If you do not write your family name last, please underline it.

535. Let the triangle ABC be isosceles with $AB = AC$. Suppose that its circumcentre is O, the D is the midpoint of side AB and that E is the centroid of triangle ACD. Prove that OE is perpendicular to CD.

536. There are 21 cities, and several airlines are responsible for connections between them. Each airline serves five cities with flights both ways between all pairs of them. Two or more airlines may serve a given pair of cities. Every pair of cities is serviced by at least one direct return flight. What is the minimum number of airlines that would meet these conditions?

537. Consider all 2×2 square arrays each of whose entries is either 0 or 1. A pair (A, B) of such arrays is *compatible* if there exists a 3×3 square array in which both A and B appear as 2×2 subarrays.

For example, the two matrices

$$\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \qquad \text{and} \qquad \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

are compatible, as both can be found in the array

$$\begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} .$$

Determine all pairs of 2×2 arrays that are not compatible.

538. In the convex quadrilateral $ABCD$, the diagonals AC and BD are perpendicular and the opposite sides AB and DC are not parallel. Suppose that the point P, where the right bisectors of AB and DC meet, is inside $ABCD$. Prove that $ABCD$ is a cyclic quadrilateral if and only if the triangles ABP and CDP have the same area.

539. Determine the maximum value of the expression

$$\frac{xy + 2yz + zw}{x^2 + y^2 + z^2 + w^2}$$

over all quartuple of real numbers not all zero.

540. Suppose that, if all planar cross-sections of a bounded solid figure are circles, then the solid figure must be a sphere.

541. Prove that the equation

$$x_1^{x_1} + x_2^{x_2} + \cdots + x_k^{x_k} = x_{k+1}^{x_{k+1}}$$

has no solution for which x_1, x_2, $\cdots$, x_k, x_{k+1} are all distinct nonzero integers.