Solutions for May

493. Prove that there is a natural number with the following characteristics: (a) it is a multiple of 2007; (b) the first four digits in its decimal representation are 2009; (c) the last four digits in its decimal representation are 2009.

Solution. First, we show that there is a positive integer n for which $n \cdot 10^4 + 2$ is a multiple of 2007. For otherwise, if none of the numbers of the form $n \cdot 10^4 + 2$ with $n = 0, 1, 2, \cdots, 2006$ were a multiple of 2007, then, by the Pigeonhole Principle, there must be two of them congruent modulo 2007. Hence, their difference, a number of the form $(k - m) \cdot 10^4$ with $0 \le k, m \le 2006$ would be a multiple of 2007, an impossibility.

Now use this number n to compose the number $M = \overline{2009007n2009}$. Since

$$\begin{aligned} M &= 2009007 \cdot 10^k + n \cdot 10^4 + 2009 \\ &= 2007000 \cdot 10^k + 2007 \cdot 10^k + 2007 + (n \cdot 10^4 + 2) \end{aligned}$$

for $5 \le k \le 8$, M is a multiple of 2007. Since it has all three of the desired characteristics, the problem is solved.

494. (a) Find all real numbers x that satisfy the equation

$$(8x - 56)\sqrt{3 - x} = 30x - x^2 - 97 \ .$$

(b) Find all real numbers x that satisfy the equation

$$\sqrt{x} + \sqrt[3]{x + 7} = \sqrt[4]{x + 80} \ .$$

Solution. (a) We must have $x \le 3$. The equation can be rewritten

$$0 = 8(x - 7)\sqrt{3 - x} + x^2 - 30x + 97 \ .$$

Let y be positive with $y^2 = 3 - x$. Then

$$\begin{aligned} 0 &= -8(y^2 + 4)y + (y^2 - 3)^2 - 30(3 - y^2) + 97 \\ &= y^4 - 8y^3 + 24y^2 - 32y + 16 = (y - 2)^4 \ . \end{aligned}$$

Hence $y = 2$, so that $x = -1$. This solution is valid.

(b) The domain of the equation is given by $x \ge 0$. One solution is $x = 1$; we prove that it is the only solution. The equation is equivalent to

$$x^2 + 4x^{3/2}(x + 7)^{1/3} + 6x(x + 7)^{2/3} + 4x^{1/2}(x + 7) + (x + 7)^{4/3} = x + 80 \ .$$

Let

$$f(x) = 4x^{3/2}(x + 7)^{1/3} + 6x(x + 7)^{2/3} + 4x^{1/2}(x + 7) + (x + 7)^{4/3}.$$

This function $f(x)$ is increasing and $f(1) = 80$. If $x > 1$, then also $x^2 > x$ and $f(x) > 80$, so that $x^2 + f(x) > x + 80$. Similarly, when $0 \le x < 1$, then $x^2 < x$ and $f(x) < 80$, so that $x^2 + f(x) < x + 80$. Hence, there is no solution to the equation save 1.

495. Let $n \ge 3$. A regular $n-$gon has area S. Squares are constructed externally on its sides, and the vertices of adjacent squares that are not vertices of the polygon are connected to form a $2n-$sided polygon, whose area is T. Prove that $T \le 4(\sqrt{3} + 1)S$. For what values of n does equality hold?

Solution. Wolog, let the sidelength of the given polygon be 1. The $2n-$sided polygon is composed of the regular $n-$gon, n squares with sidelength 1 and n isosceles triangles with equal sides of length 1 and angle between these sides equal to

$$2\pi - \left(\frac{\pi}{2} + \frac{\pi}{2} + \frac{(n-2)\pi}{n} \right) = \frac{2\pi}{n} \ .$$

Therefore

$$T = S + n + n \cdot \frac{1}{2} \sin \frac{2\pi}{n} \ .$$

On the other hand, S is the sum of the areas of n isosceles triangles, each with base 1, apex angle $2\pi/n$ and height $(1/2)\cot(\pi/n)$. Hence $S = (n/4)\cot(\pi/n)$, so that $n = 4S\tan(\pi/n)$. Therefore

$$T = S + n + n \cdot \frac{1}{2} \sin \frac{2\pi}{n} = S\left(1 + 4\tan\frac{\pi}{n} + 2\tan\frac{\pi}{n}\sin\frac{2\pi}{n} \right) \ .$$

Apply $\sin 2\theta = 2\tan\theta/(1+\tan^2\theta)$ to $\sin(2\pi/n)$ to obtain that

$$\begin{aligned} T &= S\left(1 + 4\tan\frac{\pi}{n} + \frac{4\tan^2\frac{\pi}{n}}{1+\tan^2\frac{\pi}{n}} \right) \\ & S\left(1 + 4\tan\frac{\pi}{n} + \frac{4\tan^2\frac{\pi}{n} + 4 - 4}{1+\tan^2\frac{\pi}{n}} \right) \\ &= S\left[5 + 4\left(\tan\frac{\pi}{n} - \frac{1}{1+\tan^2\frac{\pi}{n}} \right) \right] \ . \end{aligned}$$

Since $n \geq 3$ and the tangent function is increasing, $0 < \tan(\pi/n) \leq \tan(\pi/3) = \sqrt{3}$. so that

$$\tan\frac{\pi}{n} - \frac{1}{1+\tan^2(\pi/n)} \leq \sqrt{3} - \frac{1}{1+(\sqrt{3})^2} = \sqrt{3} - \frac{1}{4} \ .$$

Therefore. $T \leq 4S(\sqrt{3}+1)$, as desired. Equality holds when $n = 3$ and the polygon is an equilateral triangle.

496. Is the hundreds digit of $N = 2^{2006} + 2^{2007} + 2^{2008}$ even or odd? Justify your answer.

Solution. Observe that

$$N = 2^{2006}(1+2+4) = 7 \cdot 2^6 \cdot 2^{2000} = 7 \cdot 2^6 \cdot (2^{20})^{100} \ .$$

However, modulo 100,

$$2^{20} = 1024^2 \equiv 24^2 = 576 \equiv 76$$

and $76^n \equiv 76$ for each positive integer n. Hence

$$N = 7 \cdot 2^6 \cdot (2^{20})^{100} \equiv 7 \cdot 64 \cdot 76 \equiv 48$$

(mod 100). Denote the hundreds digit of N by h. Since N is a multiple of 8, the three digit number $h48$ must be a multiple of 8 as well. This is possible only if h is even.

Thus, the hundreds digit of the number N is even.

497. Given $n \geq 4$ points in the plane with no three collinear, construct all segments connecting two of these points. It is known that the length of each of these segments is a positive integer. Prove that the lengths of at least $1/6$ of the segments are multiples of 3.

Solution. First, we prove a lemma: *Four points are given in the plane with no three collinear. The length of each of the segments joining two of these points is an integer. Therefore, at least one of the segments has a length divisible by 3.* Denote the four points by A, B, C, D; wolog, assume that $\angle BAD = \angle BAC + \angle CAD$. Let $\angle BAC = \alpha$, $\angle CAD = \beta$ and $\angle BAD = \gamma$, so that $\gamma = \alpha + \beta$. Applying the Law of Cosines to triangles ABC, ACD and ABD, we find that

$$BC^2 = AB^2 + AC^2 - 2AB \cdot AC \cdot \cos\alpha$$

$$CD^2 = AD^2 + AC^2 - 2AD \cdot AC \cdot \cos\beta$$

and

$$BD^2 = AB^2 + AD^2 - 2AB \cdot AD \cdot \cos\gamma \ .$$

Assume, if possible, that the lengths of all six segments AB, AC, AD, BC, BD, CD are not multiples of 3. Then

$$AB^2 \equiv AC^2 \equiv AD^2 \equiv BC^2 \equiv BD^2 \equiv CD^2 \equiv 1$$

modulo 3, from which it follows that

$$2AB \cdot AC \cdot \cos\alpha \equiv 2AD \cdot AC \cdot \cos\beta \equiv 2AB \cdot AD \cdot \cos\gamma \equiv 1$$

modulo 3. Therefore

$$AC^2 \cdot AB \cdot AD \cdot \cos\alpha\cos\beta \equiv (2AB \cdot AC \cdot \cos\alpha)(2AD \cdot AC \cdot \cos\beta) \equiv 1$$

modulo 3.

¿From the foregoing equations, each of $\cos\alpha$, $\cos\beta$ and $\cos\gamma$ are rational. Let $\cos\alpha = p/q$ and $\cos\beta = r/s$, in lowest terms, where p, q, r, s are integers. None of these four integers can be multiples of 3. The denominators q and s cannot be multiples of 3 for they must cancel into side lengths, and the numerators p and r cannot be multiples of 3 since the terms containing the cosines are not divisible by 3. Hence $p^2 \equiv q^2 \equiv r^2 \equiv s^2 \equiv 1 \pmod 3$.

Since $\cos\gamma = \cos\alpha\cos\beta - \sin\alpha\sin\beta$, we have, from $2AB \cdot AD\cos\gamma \equiv 1$ and $AC^2 \equiv 1$, modulo 3, that

$$2AC^2 \cdot AB \cdot AD \cdot \cos\gamma \equiv 1 \Longleftrightarrow$$

$$2AC^2 \cdot AB \cdot AD \cdot \cos\alpha \cdot \cos\beta - 2AC^2 \cdot AB \cdot AD \cdot \frac{\sqrt{q^2-p^2}}{q} \cdot \frac{\sqrt{s^2-r^2}}{s} \equiv 1 \ .$$

The second product on the left side is a multiple of 3, so that

$$2AC^2 \cdot AB \cdot AD \cdot \cos\alpha\cos\beta \equiv 1$$

. This contradicts an earlier statement and establishes the lemma. ♣

Let $n \geq 4$. There are $\binom{n}{4}$ sets of four points, so by the lemma, there are at least this many segments whose lengths are multiples of 3, counting multiplicity (some counted more than once). Since each of the segments is counted at most $\binom{n-2}{2}$ times (for the sets of four points containing the endpoints of the segment), it follows that there are at least $\binom{n}{4}/\binom{n-2}{2}$ distinct segments whose lengths are multiples of 3.

Since

$$\frac{\binom{n}{4}}{\binom{n-2}{2}} = \frac{2n(n-1)(n-2)(n-3)}{4!(n-2)(n-3)} = \frac{1}{6} \cdot \frac{n(n-1)}{2} = \frac{1}{6} \cdot \binom{n}{2} \ ,$$

there are at least as many segments with lengths divisible by 3 as one-sixths of the number of pairs of segments, and the result follows.

498. Let a be a real parameter. Consider the simultaneous sytem of two equations:

$$\frac{1}{x+y} + x = a - 1 \; ; \tag{1}$$

$$\frac{x}{x+y} = a - 2 \; . \tag{2}$$

(a) For what value of the parameter a does the system have exactly one solution?

(b) Let $2 < a < 3$. Suppose that (x, y) satisfies the sytem. For which value of a in the stated range does $(x/y) + (y/x)$ reach its maximum value?

Solution. From the identification of the coefficients of a quadratic in terms of the sum and product of the roots, we see that $1/(x + y)$ and x are the solutions of the quadratic equation

$$0 = t^2 - (a-1)t + (a-2) = (t - \overline{a-2})(t-1) \; .$$

There are two options.

Option 1. $1/(x + y) = a - 2$, $x = 1$, so that

$$(x, y) = \left(1, \frac{3-a}{a-2}\right)$$

with $a \neq 2$.

Option 2. $1/(x + y) = 1$, $x = a - 2$, so that

$$(x, y) = (a - 2, 3 - a) \; .$$

(a) For the system to have exactly one solution, either the two options produce the same pair or only one of the options is possible. In the first instance, we have $a = 3$ and the unique solution $(x, y) = (1, 0)$ and in the second, we have $a = 2$ and the unique solution is $(x, y) = (0, 1)$.

(b) When $2 < a < 3$, both x/y and y/x are positive for either solution of the system. By the Arithmetic-Geometric Means Inequality,

$$\frac{x}{y} + \frac{y}{x} \geq 2$$

with equality if and only if $x/y = y/x$. The condition for equality is equivalent to $(a - 2)/(3 - a) = (3 - a)/(a - 2)$, or $a = 5/2$. Thus, $(x/y) + (y/x)$ attains its minimum value of 2 when $a = 5/2$.

499. The triangle ABC has all acute angles. The bisector of angle ACB intersects AB at L. Segments LM and LN with $M \in AC$ and $N \in BC$ are constructed, perpendicular to the sides AC and BC respectively. Suppose that AN and BM intersect at P. Prove that CP is perpendicular to AB.

Solution 1. Let m be a line through C parallel to AB and let AN and BM intersect m at F and E, respectively. Let CP and AB intersect at D. Triangles ADP and FCP are similar, as are triangles DBP and CEP. Hence

$$AD : CF = PD : PC = DB : CE \; .$$

Therefor $AD : BD = CF : CE$.

On the other hand, triangles ABM and CEM are similar, and triangles ABN and FCN are similar. Therefore, $AM : MC = AB : CE$ and $BN : CN = AB : CF$. However, right triangles CLM and CLN are congruent and $CM = CN$, so that $AM : BN = CF : CE$. Together with $AD : BD = CF : CE$, this yields $AD : BD = AM : BN$ and $AM : AD = BN : BD$.

Let the altitude from C to AB intersects AB at a point H. Since triangles ALM and ACH are similar, $AL : AC = AM : AH$. Similarly, from the similarity of triangles BLN and BCH, $BL : BC = BN : BH$. By the angle-bisector theorem, $AL : AC = BL : BC$. It follows that $AM : AH = BN : BH$.

Since $AM : AD = BN : BD$ and $AM : AH = BN : BH$, it follows that D and H divide AB internally in the same ratio, and so $D = H$. Thus, $CP \perp AB$ and the statement is established.

Solution 2. [J. Kileel] Use the same notation as in Solution 1. Let H be the foot of the perpendicular from C to AB. It suffices to show that AN, BM and CH are concurrent. By Ceva's Theorem, this is equivalent to showing that

$$\frac{AM}{MC} \cdot \frac{CN}{NB} \cdot \frac{BH}{HA} = 1 \ .$$

Triangles MCL and NCL are congruent (ASA), so that $CM = CN$. Triangles ALM and ACH are similar, so that $AM : HA = LM : HC$. Likewise, triangles BLN and BCH are similar; therefore,

$$BH : NB = HC : LN = HC : LM \ .$$

It follows that

$$\frac{AM}{MC} \cdot \frac{CN}{NB} \cdot \frac{BH}{HA} = \frac{AM}{HA} \cdot \frac{BH}{NB} \cdot \frac{CN}{MC} = \frac{LM}{HC} \cdot \frac{HC}{LM} \cdot \frac{CN}{MC} = 1 \ .$$