Solutions for June

500. Find all sets of distinct integers $1 < a < b < c < d$ for which $abcd - 1$ is divisible by $(a-1)(b-1)(c-1)(d-1)$.

Solution. Let $x = abcd - 1$ and $y = (a-1)(b-1)(c-1)(d-1)$. Suppose that y divides x. If any one of a, b, c, d is even, then x is odd and so y must be odd and all of a, b, c, d are even. Thus, a, b, c, d are all of the same parity. Note that

$$\frac{x}{y} < \frac{abcd}{(a-1)(b-1)(c-1)(d-1)} = \left(\frac{a}{a-1}\right)\left(\frac{b}{b-1}\right)\left(\frac{c}{c-1}\right)\left(\frac{d}{d-1}\right) .$$

Observe that, if $a \geq 5$, then

$$\frac{x}{y} < \left(\frac{5}{4}\right)\left(\frac{6}{5}\right)\left(\frac{7}{6}\right)\left(\frac{8}{7}\right) = 2 ,$$

and there are no possibilities. If $a = 4$, x is odd, but

$$\frac{x}{y} < \left(\frac{4}{3}\right)\left(\frac{6}{5}\right)\left(\frac{8}{7}\right)\left(\frac{10}{9}\right) < 3 ,$$

and again there are no possibilities. If $a = 3$, then

$$\frac{x}{y} < \left(\frac{3}{2}\right)\left(\frac{5}{4}\right)\left(\frac{7}{6}\right)\left(\frac{9}{8}\right) < 3 ,$$

and so $x/y = 2$. Since $x = 2y$ and x is not divisible by 3, then neither is y, so that none of $b-1$, $c-1$, $d-1$ is a multiple of 3. Therefore, if $b \neq 5$, then $b \geq 9$ and

$$\frac{x}{y} < \left(\frac{3}{2}\right)\left(\frac{9}{8}\right)\left(\frac{11}{10}\right)\left(\frac{15}{14}\right) < 2 ,$$

and there are no possibilities. Therefore, when $a = 3$, $b = 5$ and we are led to the equation $15cd - 1 = 2y = 16(c-1)(d-1)$, which is equivalent to

$$(c-16)(d-16) = 239 .$$

Since 239 is prime, the solution is unique: $(c, d) = (17, 255)$. We obtain the solution

$$(a, b, c, d) = (3, 5, 17, 255) .$$

Finally, suppose that a, b, c, d are all even and x/y is odd. Since

$$\frac{x}{y} < \left(\frac{2}{1}\right)\left(\frac{4}{3}\right)\left(\frac{6}{5}\right)\left(\frac{8}{7}\right) < 4 ,$$

and we find that $x/y = 3$. Thus, none of b, c, d is divisible by 3. If $b \geq 4$, then

$$\frac{x}{y} < \left(\frac{2}{1}\right)\left(\frac{8}{7}\right)\left(\frac{10}{9}\right)\left(\frac{14}{13}\right) < 3 ,$$

and there are no possibilities. Then $b = 4$ and we obtain that $8cd - 1 = 3y = 9(c-1)(d-1)$, or

$$(c-9)(d-9) = 71 .$$

As 71 is prime, the solution is unique: $(c, d) = (10, 80)$. Thus, we obtain the second and final solution

$$(a, b, c, d) = (2, 4, 10, 80) \ .$$

501. Given a list of $3n$ not necessarily distinct elements of a set S, determine necessary and sufficient conditions under which these $3n$ elements can be divided into n triples, none of which consist of three distinct elements.

Solution. A necessary and sufficient condition is that at most n of the $3n$ elements in the list occur an odd number of times.

Suppose that this condition is satisfied. Distribute one of each element appearing an odd number of times into distinct triples. Then each of the remaining elements in S occurs an even number of times, so we can sort them into at least n equal pairs (not necessarily all distinct). Sort one of these pairs into each triple, and then (if necessary) fill up the triples in any way with what is left over. Then each triple contains a pair of like elements.

On the other hand, suppose that we have distributed the $3n$ elements as specified. Then there must be a least n pairs of like elements (again, not necessarily distinct pairs). There are n elements not in these pairs, and these are the only possibilities for elements that occur an odd number of times. The result follows.

502. A set consisting of n men and n women is partitioned at random into n disjoint pairs of people. What are the expected value and variance of the number of male-female couples that result? (The *expected value* E is the average of the number N of male-female couples over all possibilities, *i.e.* the sum of the numbers of male-female couples for the possibilities divided by the number of possibilities. The *variance* is the average of the difference $(E - N)^2$ over all possibilities, *i.e.* the sum of the values of $(E - N)^2$ for the possibilities divided by the number of possibilities.)

Comment. The answer is

$$E(X) = \frac{n^2}{2n-1} \qquad \text{Var } (X) = \frac{2n^2(n-1)^2}{(2n-1)^2(2n-3)}$$

where X is the number of man-woman matches. The solution relies on some statistical theory and is given in *American Mathematical Monthly* 107 (1998), 866-867.

A direct assault seems difficult. There are

$$u_n = \frac{\binom{2n}{2}\binom{2n-2}{2}\cdots\binom{4}{2}\binom{2}{2}}{n!} = \frac{(2n)!}{2^n n} = (2n-1)(2n-3)\cdots(3)(1)$$

ways of pairing off the $2n$ people. The number of men not paired off with women must be even, as they are paired with each other, and similarly for the women. Suppose that we want $n - 2k$ man-woman pairs. There are $\binom{n}{2k}$ ways of picking the men to be paired off with each other and so $\binom{n}{2k}u_k$ of selecting and pairing them off. Similarly, there are $\binom{n}{2k}u_k$ of selecting and pairing off the $2k$ women not in the couples. As for the $n-2k$ men to be paired off with women, there are $(n - 2k)!$ ways of pairing them off with the women not paired with other women. Therefore the number of ways of pairing so that there are exactly $n - 2k$ man-woman couples is

$$\left[\binom{n}{2k}u_k\right]^2 (n-2k)!$$

and the average number of couples over all the pairings is

$$\left[\binom{n}{2k}u_k\right]^2 (n-2k)!(n-2k) \div u_n \ .$$

503. A natural number is *perfect* if it is the sum of its proper positive divisors. Prove that no two consecutive numbers can both be perfect.

Solution. We review basic information about the sum of divisors function. For any positive integer n, the function $\sigma(n)$ is the sum of all the positive divisors of n, including both 1 and n. For $n \geq 2$, $\sigma(n) \geq n+1$ with equality if and only if n is prime. If m and n have greatest common divisor equal to 1, then $\sigma(mn) = \sigma(m)\sigma(n)$. For any prime p and positive integer exponent c, we have that $\sigma(p^c) = (p^{c+1}-1)/(p-1)$. A positive integer n is perfect if and only if $\sigma(n) = 2n$.

Lemma. (Euclid-Euler) An even positive integer r is perfect if and only if it is of the form $r = 2^{p-1}(2^p-1)$ where both p and $2^p - 1$ are prime.

Proof of Lemma. Let n be an even perfect number. Then $n = 2^k m$ where $k \geq 1$ and m is odd, so that

$$2^{k+1}m = 2n = \sigma(n) = \sigma(2^k)\sigma(m) = (2^{k+1} - 1)\sigma(m) \ .$$

Since the greatest common divisor of 2^{k+1} and $2^{k+1} - 1$ is 1, there exists a positive integer w for which $m = (2^{k+1} - 1)w$ and $\sigma(m) = 2^{k+1}w$.

Suppose, if possible, that $w > 1$. Then the divisors of m include the distinct $1, w, m$ so that

$$\sigma(m) \geq 1 + w + m = 1 + 2^{k+1}w > \sigma(m) \ ,$$

an impossibility. Therefore $w = 1$ and $\sigma(m) = 2^{k+1} = m + 1$, so that $m = 2^{k+1} - 1$ is prime. This forces also $k + 1$ to be prime (why?) and the necessity of the condition follows. It is straightforward to verify that each number of the given form is perfect. ♣

Since 5 and 7 are not perfect numbers, we need not consider the case where the even number of the pair is $6 = 2(2^2 - 1)$. We obtain the desired result by showing that for any odd prime p, neither of the numbers $u = 2^{p-1}(2^p - 1) - 1$ and $v = 2^{p-1}(2^p - 1) + 1$ is perfect.

Observe that

$$2^{p-1}(2^p - 1) = 2 \times 4^{p-1} - 4^{(p-1)/2} \equiv 2 \times 4 - 4 = 4$$

modulo 12, since every positive integer power of 4 is congruent to 4 modulo 12. Hence $u \equiv 3$ and $v \equiv 5$ (mod 12), and so neither u nor v are squares. Hence each has an even number of divisors, that can be paired off as $(d, u/d)$ and $(d, v/d)$ respectively, where d is less than the square root of u and v respectively.

We have that $u \equiv -1 \pmod 4$, so that, as $d(u/d) = u \equiv -1$, $\{d, u/d\} \equiv \{1, -1\}$ and $d + (u/d) \equiv 0 \pmod 4$. Hence

$$\sigma(u) = \sum \{d + (n/d) : d|u, d < \sqrt{u}\} \equiv 0$$

modulo 4, while $2u \equiv 6 \pmod 4$. Hence $\sigma(u) \neq 2u$ and u is not perfect.

We have that $v \equiv -1 \pmod 3$, so that $\{d, v/d\} \equiv \{1, -1\}$ and $d + (v/d) \equiv 0 \pmod 3$, for every divisor d of v. Hence

$$\sigma(v) = \sum \{d + (n/d) : d|v, d < \sqrt{v}\} \equiv 0$$

modulo 3, while $2v \equiv 10 \pmod{12}$. Hence $\sigma(v) \neq 2v$ and v is not perfect.

504. Find all functions f taking the real numbers into the real numbers for which the following conditions hold simultaneously:

(a) $f(x + f(y) + yf(x)) = y + f(x) + xf(y)$ for every real pair (x, y);

(b) $\{f(x)/x : x \neq 0\}$ is a finite set.

Solution. The function f must be the identity function $f(x) = x$ for all x.

By (b), there exists a number k such that the set $S_k = \{x : f(x) = kx\}$ is infinite. From (a), for all $x \in S_k$,

$$f(x + kx + kx^2) = f(x + f(x) + xf(x)) = x + kx + kx^2 \ .$$

Hence S_1 has infinitely many elements.

Suppose, if possible, that there exists y such that $f(y)/y = m \neq 1$. Then, for all $x \in S_1$,

$$f(x + my + yx) = f(x + f(y) + yf(x)) = y + f(x) + xf(y) = y + x + mxy \ .$$

As x ranges over the infinite set S_1, by (b), it is not possible that

$$\frac{f(x + my + yx)}{x + my + yx} = \frac{x(1 + my) + y}{x(1 + y) + my}$$

takes infinitely many values. Hence, there are x_1 and x_2 in S for which $x_1 \neq x_2$ and

$$\frac{f(x_1 + my + yx_1)}{x_1 + my + yx_1} = \frac{f(x_2 + my + yx_2)}{x_2 + my + yx_2} \ .$$

Then,

$$[x_1(1 + my) + y][x_2(1 + y) + my] = [x_2(1 + my) + y][x_1(1 + y) + my]$$

so that, since $(x_1 - x_2)y \neq 0$,

$$(x_1 - x_2)my(1 + my) = (x_1 - x_2)y(1 + y) \Leftrightarrow (m^2 - 1)y + (m - 1) = 0 \Leftrightarrow (m + 1)y + 1 = 0 \ .$$

Thus, $m = -(y + 1)/y$, so that $f(y) = my = -(y + 1)$. This means that y is uniquely determined by m and that S_m is a singleton.

For all $x \in S_1$, by condition (i), we find that

$$\begin{aligned} f((y + 1)(x - 1)) &= f(x - (y + 1) + yx) = f(x + f(y) + yf(x)) = y + f(x) + xf(y) \\ &= y + x - x(y + 1) = -y(x - 1) = \frac{-y}{y + 1}(y + 1)(x - 1) \\ &= \frac{-1}{m}(y + 1)(x - 1) \ . \end{aligned}$$

But $\{(y + 1)(x - 1) : x \in S_1\}$ is an infinite set, so $S_{-1/m}$ must be infinite.

However, we can go through the foregoing argument with m replaced by $-1/m$ and deduce, either that $f(x)/x$ takes infinitely many values or that $S_{-1/m}$ is a singleton, both of which yield constradictions. The result follows.

505. What is the largest cubical present that can be completely wrapped (without cutting) by a unit square of wrapping paper?

Solution. The side of the largest possible cubical present that can be wrapped is $\sqrt{2}/4$.

Let s be the side length of the cube. For any point on the cube, there is another point whose distance away is at least $2s$, so that for each point on the square, there is a point whose distance away from it is at least $2s$. This is true in particular for the centre of the square, so the diagonal of the square is at least $4s$. Hence $\sqrt{2} \geq 4s$, so that $s \leq \sqrt{2}/4$.

On the other hand, if we have a cube of this size, we can place it right in the centre of the wrapping square with its sides parallel to the diagonals of the square, and fold the corners of the square over the lateral faces of the cube with them meeting in the middle of the top face of the cube.

506. A two-person game is played as follows. A position consists of a pair (a, b) of positive integers. Playes move alternately. A move consists of decreasing the larger number in the current position by any positive multiple of the smaller number, as long as the result remains positive. The first player unable to make

a move loses. (This happens, for example, when $a = b$.) Determine those positions (a, b) from which the first player can guarantee a win with optimal play.

Solution. Let $\phi = \frac{1}{2}(1 + \sqrt{5})$, the larger root of the polynomial $x^2 - x - 1$. Note that ϕ is irrational, that $1/\phi = \phi - 1$ and that $0 < 1/\phi < 1 < \phi < 2$. We show that the first player can guarantee a win if and only if the ratio of the larger to the smaller number exceeds ϕ.

Let $\mathfrak{W}$ be the set of pairs $\{a, b\}$ with $a > \phi b$ and $\mathfrak{L}$ the set of pairs with $b < a < \phi b$. The result will follow if we prove that, from any pair in $\mathfrak{W}$, the next player can leave a pair in $\mathfrak{L}$ and that, from any pair in $\mathfrak{L}$, the next player must go to a pair in $\mathfrak{W}$.

Suppose that $\{a, b\} \in \mathfrak{W}$. Then $a/b > \phi$ and there is a (unique) poaitive integer k for which $\phi - 1 < (a/b) - k < \phi$. Choosing this k yields the pair $\{b, a - bk\}$. Since

$$\frac{b}{a - bk} = \frac{1}{(a/b) - k} < \frac{1}{\phi - 1} = \phi \ ,$$

the pair $\{b, a - bk\} \in \mathfrak{L}$.

On the other hand, if $\{a, b\} \in \mathfrak{L}$, then $b < a < \phi b$, so that $a < 2b$. The only legal move leads to the pair $\{b, a - b\}$. But then

$$\frac{b}{a - b} = \frac{1}{(a/b) - 1} > \frac{1}{\phi - 1} = \phi \ ,$$

so that $\{a, a - b\} \in \mathfrak{W}$.

Therefore, the first player has a winning strategy if and only if presented with a pair $\{a, b\}$ with $a > \phi b$.