Solutions for July.

507. Prove that, if a, b, c are positive reals, then

$$\log^2 \frac{ab}{c} + \log^2 \frac{bc}{a} + \log^2 \frac{ca}{b} + \frac{3}{4} \geq \log(abc) \ .$$

Solution 1. [A. Dhawan] By the Cauchy-Schwarz Inequality,

$$\left(\log^2 \frac{ab}{c} + \log^2 \frac{bc}{a} + \log^2 \frac{ca}{b}\right)(1^2 + 1^2 + 1^2) \geq \left(\log \frac{ab}{c} + \log \frac{bc}{a} + \log \frac{ca}{b}\right)^2 = \log^2(abc) \ ,$$

whence

$$\left(\log^2 \frac{ab}{c} + \log^2 \frac{bc}{a} + \log^2 \frac{ca}{b}\right) \geq \frac{1}{3}\log^2(abc) \ .$$

Also,

$$\begin{aligned}\frac{1}{3}\log^2(abc) + \frac{3}{4} - \log(abc) &= \frac{1}{12}(4\log^2(abc) - 12\log(abc) + 9) \\ &= \frac{1}{12}(2\log(abc) - 3)^2 \geq 0 \ .\end{aligned}$$

Hence

$$\log^2 \frac{ab}{c} + \log^2 \frac{bc}{a} + \log^2 \frac{ca}{b} + \frac{3}{4} \geq \frac{1}{3}\log^2(abc) + \frac{3}{4} \geq \log(abc) \ .$$

Solution 2. The inequality is equivalent to

$$(\log a + \log b - \log c)^2 + (\log b + \log c - \log a)^2 + (\log c + \log a - \log b)^2 + \frac{3}{4} \geq \log a + \log b + \log c \ .$$

Denoting $\log a + \log b - \log c = x$, $\log b + \log c - \log a = y$ and $\log c + \log a - \log b = z$, we get that $x + y + z = \log a + \log b + \log c$.

Since $(x - \frac{1}{2})^2 + (y - \frac{1}{2})^2 + (z - \frac{1}{2})^2 \geq 0$, it follows that $x^2 + y^2 + x^2 + \frac{3}{4} \geq x + y + z$, which is the desired inequality. Equality holds if and only if $x = y = z = \frac{1}{2}$, which requires that $a = b = c$.

508. Let a, b, c be integers exceeding 1 for which both $\log_a b + \log_b a$ and $\log_a^2 b + \log_b^2 a$ are rational. Prove that, for every positive integer n, $\log_a^n b + \log_b^n a$ is rational.

Solution. The result holds for $n = 0, 1, 2$. Assume, as an induction hypothesis, that $\log_a^k + \log_b^k$ is rational for $0 \leq k \leq n$.

Taking $x = \log_a b$ and $y = \log_b a$ in the identity

$$x^{k+1} + y^{k+1} = (x + y)(x^k + y^k) - xy(x^{k-1} + y^{k-1}) \ ,$$

and noting that $xy = 1$, we find that, by induction, $\log_a^{k+1} b + \log_b^{k+1} a$ is rational. The result follows.

509. Let $ABCDA'B'C'D'$ be a cube where the point O is the centre of the face $ABCD$ and $|AB| = 2a$. Calculate the distance from the point B to the line of intersection of the planes $A'B'O$ and $ADD'A'$ and the distance between AB' and BD. (AA', BB', CC', DD' are edges of the cube.)

Solution. Let M be the intersection of the plane $A'B'O$ and the line AD. Then M is the midpoint of DA. Therefore, the intersection between the planes $A'B'O$ and $ADD'A'$ is the line $A'M$. Hence, the distance from the point B to the line of intersection of the planes $A'B'O$ and $ADD'A'$ is the height of the isosceles triangle $BA'M$ from B. Since $|MB| = |A'M| = a\sqrt{5}$ and $|BA'| = 2a\sqrt{2}$, we get the height $2a\sqrt{6}/\sqrt{5}$.

For the distance between AB' and BD, we note that BD and AB' are included in the parallel planes $C'DB$ and $AB'D'$, respectively. Therefore, the distance between BD and AB' is the distance between the planes $C'DB$ and $AB'D'$, namely $(1/3)|AC'| = 2a\sqrt{3}/3$.

510. Solve the equation

$$\sqrt[3]{x^2+2} + \sqrt[3]{4x^2+3x-2} = \sqrt[3]{3x^2+x+5} + \sqrt[3]{2x^2+2x-5} \ .$$

Solution. Let the four members of the equation be denoted by m, n, p, q respectively, so that $m+n = p+q$. Also

$$(m+n)^3 - 3mn(m+n) = m^3 + n^3 = p^3 + q^3 = (p+q)^3 - 3pq(p+q) \ ,$$

whence $mn(m+n) = pq(p+q)$.

If $m + n = p + q = 0$, then $5x^2 + 3x = 0$, so that $x = 0$ or $x = -3/5$. Both of these work.

If $m + n = p + q \neq 0$, then $mn = pq$. Thus

$$\begin{aligned} 0 &= (3x^2+x+5)(2x^2+2x-5) - (x^2+2)(4x^2+3x-2) \\ &= 2x^4 + 5x^3 - 9x^2 - x - 21 = (x^2+2x-7)(2x^2+x+3) \ , \end{aligned}$$

so that $x = -1 \pm 2\sqrt{2}$. The second quadratic factor has nonreal roots. Therefore, the solution set of the equation is

$$\{-1-2\sqrt{2}, -3/5, 0, -1+2\sqrt{2}\} \ .$$

511. Find the sum of the last 100 digits of the number

$$A = 1 \cdot 2 \cdot 3 \cdot \ldots \cdot 2005 \cdot 2006 + 2007 \ .$$

Solution. The exponent of the power of 5 that divides 2006! is equal to

$$\left\lfloor \frac{2006}{5} \right\rfloor + \left\lfloor \frac{2006}{5^2} \right\rfloor + \left\lfloor \frac{2006}{5^3} \right\rfloor + \left\lfloor \frac{2006}{5^4} \right\rfloor = 401 + 80 + 16 + 3 = 500 \ .$$

The exponent of the power of 2 that divides 2006! is at least this big, so that the decimal expansion of 2006! ends in 500 zeros. Hence the sum of the last 100 digits of the given expression is $2 + 0 + 0 + 7 = 9$.

512. Prove that

$$\binom{3n}{n} = \sum_{k=0}^{n} \binom{2n}{k}\binom{n}{k}$$

when $n \geq 1$.

Solution. The result is obtained by comparing the coefficients of x^n when both sides of the equation

$$(1+x)^{3n} = (1+x)^{2n}(1+x)^n$$

are explanded binomially.

513. Solve the equation

$$2^{1-2\sin^2 x} = 2 + \log_2(1 - \sin^2 x) \ .$$

Solution. Let $t = \cos 2x$. The equation becomes $2^t = 1 + \log_2(1+t)$. For $0 < t < 1$, we have that

$$2^t < t + 1 < 1 + \log_2(1+t) \ ,$$

and for $-1 \leq t < 0$, we have that

$$2^t > t + 1 > 1 + \log_2(1+t) \ .$$

(Sketch the graphs of 2^t and $t + 1$.) Therefore $t = 0$ or $t = 1$ and so $x = 0$ or $x = \pi/4$.